WORKSHEET 15

Vertices, Edges, and Faces

The data of a graph includes only vertices and edges. However, if we draw a graph on the plane with no edges crossing over other edges (i.e. a *planar graph*), we can also consider what is left of the plane once we remove the graph. The regions left over once we remove the graph are called the *faces*. We include the outer region as one of the faces. For instance, the planar graph in Figure 1 has 6 faces, as marked.

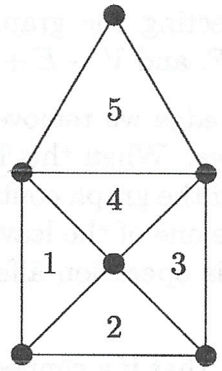


Figure 1. A planar graph with faces marked.

Problem 15.1. Draw several planar graphs and count the number V of vertices, E of edges, and F of faces in each. Do you notice any patterns?

As you probably noticed (spoiler alert!), it appears that if a planar graph is connected, then $V - E + F = 2$. We will now explain this fact.

Problem 15.2. Given a graph with one vertex and no edges, compute the values of V, E, F, and $V - E + F$.

Now, let's imagine adding edges to this graph, one at a time, seeing how this affects the value of $V - E + F$. We will assume throughout the rest of the worksheet that all graphs under consideration are connected and planar.

Problem 15.3. Suppose we add an edge between two existing vertices that don't already share an edge, without adding crossings. How does this operation affect the values of V, E, F, and $V - E + F$?

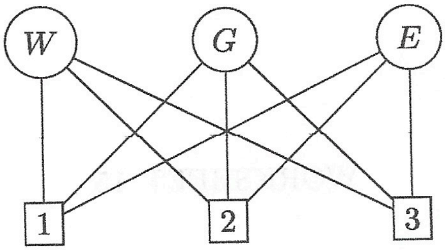


Figure 2. Connecting houses to utilities.

Problem 15.4. Suppose we add a new vertex to the graph. To keep the graph connected, let's connect it to one other vertex with an edge. How does this operation affect the values of V, E, F, and $V - E + F$?

Now, let's go the other way, removing edges one by one.

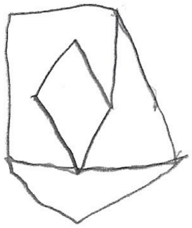
Problem 15.5. Suppose we start with a planar graph and remove an edge without disconnecting the graph. How does this operation affect the values of V, E, F, and $V - E + F$?

Problem 15.6. The edge we remove might be the only edge adjacent to one of its vertices. When this happens, we also remove this vertex, known as a *leaf*. (If the graph contains just two vertices and one edge, then we only remove one of the leaves, so that the graph remains connected.) How does this operation affect the values of V, E, F, and $V - E + F$?

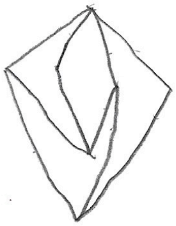
Problem 15.7. Show that if a connected planar graph has at least two vertices, then there exists some edge that we can remove as in the previous two problems without disconnecting the graph.

Problem 15.8. Show that $V - E + F = 2$ for any connected planar graph.

Let's now see a few classic applications of this formula.



Problem 15.9. Suppose we have three houses, and we wish to connect each of them to the water, gas, and electricity providers, with a separate line from each house to each company. Is there a way to make all nine connections without any of these lines crossing over any others? Figure 2 shows a way to do it with several crossings. We call this graph $K_{3,3}$.



Problem 15.10. Is there a planar graph with 5 vertices, with an edge between every pair of vertices? We call this graph K_5.

Problem 15.11. Show that if $G = (V, E)$ is a connected planar graph, then $E \leq 3V - 6$.

DEFINITION. The *degree* $\deg(v)$ of a vertex v of a graph is the number of edges containing it.

PROBLEM 15.12. Show that if G is a planar graph, then it has a vertex v with $\deg(v) \leq 5$.

If we take a plane and bunch it up so that the points far away meet at a point at ∞, then we end up with a (possibly distorted) sphere, so we can think of a sphere as being a plane together with an extra point at ∞. Thus any planar graph can be considered instead as a spherical graph, and again we have $V - E + F = 2$. For any spherical graph, we can think of it as being a planar graph by making sure that the point at ∞ is in the interior of some face, after which that face becomes the outer face of the corresponding planar graph.

Now, if we take the outside of any convex polyhedron, then we can inflate it to a sphere, in which case we have a spherical graph.

PROBLEM 15.13. Use the formula $V - E + F = 2$ for spherical graphs to find an expression for V, E, F for Platonic solids in terms of the Schläfli symbols $\{p, q\}$.

PROBLEM 15.14. Suppose we have a convex polyhedron made up of only pentagons and hexagons. How many pentagons must it contain?

Vertices, Edges, and Faces

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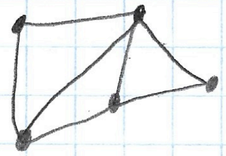
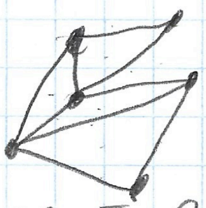
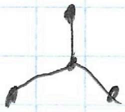
15.1:



$V=5, E=7, F=4.$



$V=6, E=9, F=5.$



$V=4, E=3, F=1.$

$V=1, E=0, F=1.$

Pattern: $5-7+4=2$, $6-9+5=2$, $4-3+1=2$, $1-0+1=2$, $\ldots$ $V-E+F=2$.

15.2: In 15.1's last example:

$V=1, E=0, F=1, V-E+F=1-0+1=2.$

15.3: Example:

$V=4, E=4, F=2.$

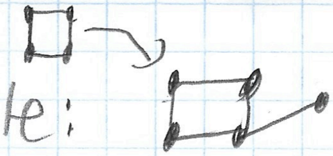
$V-E+F=4-4+2=2.$

$V=4, E=5, F=3.$

$V-E+F=4-5+3=2.$

When one connection is made, E grows by 1 and one face is split in two to grow F by 1.

$V-(E+1)+(F+1)=V-E-1+F+1=V-E+F$, so the equation does not change.

15.4: Again, example: 
$V=5, E=5, F=2, V-E+F=2$.
Adding a separate edge-vertice connection adds 1 to both V and E. Again, $(V+1)-(E+1)+F=V+1-E-1+F=V-E+F$. The equation does not change.

15.5: Starting with the 15.3/4 example:
 $V=4, E=3, F=1, V-E+F=2$.
Removing an edge also removes a face, so the equation $V-E+F$ remains constant.

15.6: Start with 15.4's example: .
Then, remove the edge: 
Back to $V=4, E=4, F=2$. Removing both an edge and a "leaf" vertice does not affect $V-E+F$:
$(v-1)-(E-1)+F=V-1-E+1+F=V-E+F$.

15.7: Case 1: The graph has a cycle. Each vertice in a cycle connects to 2 others, so removing this cycle does not cause a disconnection.... (cont.)

15.7(cont.): ... Case 2: The graph is a tree (has no cycles.) In every tree graph (of size >2 vertices) there are always at least two leaf vertices, so any leaf vertice and its connecting edge can be removed without disconnection.

15.8: It has been shown that any addition or removal of any type of edge does not change $V-E+F$, and cannot disconnect the graph. Therefore, because in a • (point) $V=1$, $E=0$, and $F=1$ to make $V-E+F=2$, any edges added or removed to or from the graph will not change the value of $V-E+F$ — always equalling ②.

15.9: Proof by Contradiction: Assume a solution exists. According to $V-E+F=2$ where $V=6$ and $E=9$, F must be 5 ... (cont.)

15.9(cont.): ... and the $K_{3,3}$ has 5 faces.

Because there are no house-to-house or utility-to-utility connections, it is impossible to form a face with 3 edges. Therefore, the 9 edges must be split up among 5 4-edged (or more) faces, a total of 20 edges with 11 overlapping. Because every edge must serve as a side of exactly no more than 2 faces, the sum of all of the edges including overlaps must be no more than 2 times the sum of all distinct edges, or $2 \cdot 9 = 18$. But the edges with overlaps count to 20, and $18 < 20$ is a contradiction. Because the 5-face-9-edge setup is impossible, the $K_{3,3}$ utilities problem is also impossible.

‡An alternate to Euler's Formula.

*No double edges, only one connection per point

15.10: In a K_5, Euler's Formula ($V-E+F=2$) with $V=5$ and $E=10$ results in $F=7$. Because there are 10 edges, each contributing to a maximum of 2 faces, there must be exactly 20 edges shared between the 7 faces. But because each face must have at least three edges in a K_5*, there must be a minimum of 21 ($=3\cdot7=$ edges per face $\cdot$ total faces) shared edges for the seven faces. Of course, 20 is not a minimum of 21, so this contradicts a K_5.

15.11: From problem 15.9, the number of shared edges is no more than twice the total edges, and each face has at least 3 edges in a planar graph. Therefore, $2E \geq 3F$ because each face must have a <u>minimum</u> of 3 edges. Then, substitute into $F=E-V+2$‡: $2E \geq 3(E-V+2)$...

..., $2E \geq 3E-3V+6$, $-E \geq -3V+6$, $\boxed{E \leq 3V-6}$.

15.12: Assume that a planar graph exists with all vertices of degree greater than 5. Then, the sum of all degrees (shared edges) would be $2E \geq 6V$ (because there are at least 6 edges for each vertice.) Therefore, $E \geq 3V$. Also, the planar graph must follow $E \leq 3V-6$. This means that $3V \leq E \leq 3V-6$. Simplify: $0 \leq E \leq -6$, $\underline{0 \leq -6.}$ Of course, this is impossible and a vertex of degree 5 or less must exist.

15.13: Note: Schläfi $\{p, q\}$ where p = number of edges per face and q = edges per vertex, for a Platonic Solid. Therefore, $p \cdot F = 2E$ (double-counting each edge) and $q \cdot V = 2E$.↑ Simplify: $F = \frac{2E}{p}$ and $V = \frac{2E}{q}$. Substitute into Euler's: $V - E + F = 2 = \frac{2E}{q} - E + \frac{2E}{p} = 2$. Factor: $E\left(\frac{2}{q} - 1 + \frac{2}{p}\right) = 2 = E\left(\frac{2p - pq + 2q}{pq}\right) = 2 \ldots$

$\ldots E = 2 \div \left(\frac{2p-pq+2q}{pq}\right) = \boxed{\frac{2pq}{2p-pq+2q}.}$ (cont.)...

15.13 (cont.): ... $V = \frac{2E}{q} = \frac{2}{\cancel{q}_1} \cdot \left(\frac{2p\cancel{q}}{2p-pq+2q}\right) = \boxed{\frac{4p}{2p-pq+2q}}$.

Finally, $F = \frac{2E}{p} = \frac{2}{\cancel{p}_1} \cdot \left(\frac{2\cancel{p}q}{2p-pq+2q}\right) = \boxed{\frac{4q}{2p-pq+2q}}$.

15.14: These polyhedra are called "Goldberg's." The number F includes (F_5, F_6) where F_i is a face with i sides. Also, each vertex has 3 edges (this is true for a dodecahedron, the most primitive Goldberg polyhedron, and cannot increase with shapes of higher side count added.) Therefore, $V = \frac{5f_5 + 6f_6}{3}$ and $E = \frac{5f_5 + 6f_6}{2}$. Substitute into Euler's Formula: $\frac{5f_5 + 6f_6}{3} - \frac{5f_5 + 6f_6}{2} + f_5 + f_6 = 2$, $10f_5 + 12f_6 - 15f_5 - 18f_6 + 6f_5 + 6f_6 = 12$...

... $= f_5 + 0 \cdot f_6 = 12$, $\boxed{f_5 = 12.}$ Therefore, there are 12 pentagonal faces on every Goldberg convex polyhedra.