WORKSHEET 2

The Thue–Morse Sequence

The Thue–Morse sequence is a remarkable sequence of 0's and 1's that arises in many apparently quite different ways. Let's start with the definition, in terms of binary numbers. Given a nonnegative integer n, we can write it in binary, in a unique way. For instance, the binary representation of 22 is 10110, which means that $22 = 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$. Given a nonnegative integer n, we define t_n to be

$$t_n = \begin{cases} 1 & \text{the binary representation of } n \text{ has an odd number of 1's,} \\ 0 & \text{the binary representation of } n \text{ has an even number of 1's.} \end{cases}$$

The sequence t_n, starting with $n = 0$, begins

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

$$0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \ldots \tag{2.1}$$

The sequence t_n is called the *Thue–Morse sequence.*

Definition. We say that n is an *evil* number if $t_n = 0$, and n is an *odious* number if $t_n = 1$.

Let's eliminate the commas from (2.1) for typographical purposes. If we cut off the sequence after $1, 2, 4, 8, 16, \ldots$ terms, we get

$$\begin{aligned} S_0 &= 0 \\ S_1 &= 01 \\ S_2 &= 0110 \\ S_3 &= 01101001 \\ S_4 &= 0110100110010110, \end{aligned}$$

and so on.

Given a sequence S of 0's and 1's, let $\overline{S}$ be the sequence obtained from S by replacing all the 0's with 1's, and all the 1's with 0's. For instance, we have

$$\overline{011010111} = 100101000.$$

Given two binary sequences S and T, let ST denote the concatenation: write down all the digits of S, followed by all the digits of T.

PROBLEM 2.1. Show that, in the decomposition above, we have $S_{n+1} = S_n\overline{S_n}$, the concatenation of S_n and $\overline{S_n}$. Write this relation in terms of various values of t_n.

PROBLEM 2.2. A sequence $S = s_0s_1\cdots s_n$ is called a *palindrome* if $s_j = s_{n-j}$ for all j, i.e. it reads the same forward and backward. Is S_n sometimes a palindrome? For which values of n? Determine all values of n for which S_n is a palindrome, and explain why your answer is correct.

PROBLEM 2.3. Start with a 0. Then, at every stage, replace each 0 with a 01 and each 1 with a 10. Write out the first five stages of this process. What do you notice? Can you explain what is going on?

PROBLEM 2.4. Suppose that, in the previous problem, you instead replace each 0 with 01, and each 1 with 0. What do you get? How many digits are there in the n^{th} stage of this process?

PROBLEM 2.5. Express t_{2n} in terms of t_n.

PROBLEM 2.6. Express t_{2n+1} in terms of t_n.

PROBLEM 2.7. What happens when we look at only the even terms of the Thue–Morse sequence? How about only the odd terms?

PROBLEM 2.8. Suppose $0 \le j < 2^n$. Express t_{2^n+j} in terms of t_j.

0 1 10 11 100 101 110

PROBLEM 2.9. Show that if $t_n = t_{n+1}$, then n must be odd. 0 1 2 3 4 5 6

PROBLEM 2.10. Show that there is no nonnegative integer n such that $t_n = t_{n+1} = t_{n+2}$.

PROBLEM 2.11. Show that among any five consecutive terms of the Thue–Morse sequence, there must be two consecutive terms that are equal.

We say that a sequence S is *overlap-free* if there is no consecutive subsequence of S of the form $cxcxc$, where c and x are nonempty sequences. For instance, the sequence

01001001010111

is not overlap-free: for instance, we can let c and x be the red and blue subsequences, respectively:

01001001010111.

PROBLEM 2.12. Show that, in order to guarantee that a sequence is overlap-free, it suffices to let c as above be a single digit.

PROBLEM 2.13. Show that the Thue–Morse sequence is overlap-free. (You will need to use the results of many of the previous problems for this.)

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PROBLEM 2.14. In CHESS, if a position is repeated three times, then either player can declare the game to be drawn. A previous version of this rule is that if a sequence of moves is repeated three times in a row, then the game can be declared drawn. Show that this previous version allows games to last forever.

The Thue-Morse Sequence

Alexander F. Mathematical Thinking 9/30/25

2.1: Let's start with $n=0$:

$S_1 = 0 = S_0\overline{S_0}$. (this is a weak beginning, but it gets stronger).

Now, if we take $n=1$:

$S_2 = S_1\overline{S_1}$. This looks like:

$n=2$: $n=3$:

0 $\xrightarrow{\text{flip it}}$ 1 $\xrightarrow{\text{put it on the sequence}}$ 01 $\xrightarrow{\text{flip it}}$ 10 $\xrightarrow{\text{put it on}}$ 0110 $\xrightarrow{\text{flip it}}$ . . .

This diagram explains $S_{n+1} = S_n\overline{S_n}$ in pictures. Eventually, if we compare this pattern to the Thue-Morse sequence, we get:

Pattern: 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 . . .
Thue-M: 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 . . .

They are the same! In fact, if you repeat this forever, you will form the Thue-Morse Sequence!

2.2: If we list out each sequence,'s: the pattern discovered is that all even S_n's are palindromes. If we take such an even S_n, such as S_{2n+2}, it $= S_{2n+1}\overline{S_{2n+1}}$.
But $S_{2n+2} = S_{2n}\overline{S_{2n}}(\overline{S_{2n}\overline{S_{2n}}})$ which reduces to $S_{2n}\overline{S_{2n}}\,\overline{S_{2n}}S_{2n}$, a palindrome.

2.3:

0

0 1

0 1|1 0

0 1 1 0 1 0 0 1

0 1 1 0 1 0 0 1|1 0 0 1 0 1 1 0

Palindromic

This also starts to form the Thue-Morse, just like the pattern in 2.1. Here, every odd sequence is a palindrome, while in 2.2 every even sequence is a palindrome. Overall, this process gives the same result as 2.1 - the Thue-Morse.

2.4:

0

0 1

0 1 0

0 1 0 0 1

0 1 0 0 1 0 1 0

This doesn't form the Thue-Morse, and just looking at the numbers in each row (1, 2, 3, 5, 8) doesn't give anything. So, we can look at the relationship of 1's and 0's.

n	0	1	2	3	4	5 (not shown)
# of 0's	1	1	2	3	5	13
# of 1's	0	1	1	2	3	8
#s in row	1	2	3	5	8	21

(cont.)

Each of these relationships has a Fibonacci-like sequence, and there is a clear reason why:

0 $\quad n=0$
0 1 $\quad n=1$
0 1 0 $\quad n=2$
0 1 0 0 1 $\quad n=3$
0 1 0 0 1 0 1 0 $\quad n=4$

Each sequence is the concatenation (sum) of the previous two, just like the Fibonacci sequence (in a way.) Therefore, where S_n denotes the sequence, $S_n = S_{n-1} + S_{n-2}$.

2.5: Because we know every T_n is in the Thue-Morse sequence, let's list them:

$(t_n =)\ n =$ 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21

0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 1 0 0 1 0 1 . . .

If $n=1$, we have a 1, and if $n=2$ we have a 1.
If $n=2$, 1, if $n=4$, 1.
If $n=3$, 0, if $n=6$, 0.
If $n=4$, 1, if $n=8$, 1.
If $n=\ldots$, t_n, if $n=\ldots$, $t_n = t_{2n}$!

This pattern shows that $t_{2n} = t_n$ in the Thue-Morse sequence.

2.6: t_{2n+1} is the opposite of t_{2n} - while t_{2n} is always $= t_n$, t_{2n+1} is always $1 - t_n$ (the opposite of t_n, 0 or 1) in the Thue-Morse.

*It doesn't matter - any n will show the pattern.

2.7: Let's compare even terms $(t_0, t_2, t_4, t_6 \ldots)$ with odd terms $(t_1, t_3, t_5, t_7 \ldots)$:

"Even" list: 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 ...
"Odd" list: 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 ...

The "even" list forms the Thue-Morse, and the "odd" list forms the exact opposite of the Thue-Morse (flip all 0's and 1's.)

2.8: We can make a table, assuming that $n=3$:*

j	1	2	3	4	5	6	7	8	9
t_j	1	1	0	1	0	0	1	1	0
t_{2^n+j} (t_{8+j})	0	0	1	0	1	1	0	1	0

The pattern appears to be the opposite $(1-t_j)$ until we reach $t_{2^n} = t_j$, at the 8th term. This makes a piecewise function:

$$t_{2^n+j} = \begin{cases} 1 - t_j & \text{when } t_j < t_{2^n} \\ t_j & \text{when } t_j \geq t_{2^n} \end{cases}$$

list out Thue-Morse and prove by binary representations

2.9: Again, the Thue-Morse has the answers...

T-M	0	1	$\overset{=}{1}$	0	1	0	$\overset{=}{0}$	1	$\overset{=}{1}$	0	$\overset{=}{0}$	1	0	1	$\overset{=}{1}$	0	1	0	$\overset{=}{0}$	1	0	1	...
#index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	...

The pattern shown here becomes clear: Even indexes never result in consecutive t_n's - only odd ones do (and not always.) An = sign marks this pattern.

This pattern can also be seen in binary:

0	1 = 10	11	100	101 = 110	
0	1 = 2	3	4	5 = 6	...

Every time the number of ones doesn't change, it's an odd index!

2.10: If we look back to 2.3, the Thue-Morse is buit by:

- making all 0's become 01,
- making all 1's become 10,
- and starting with a 0.

These "rules" mean that the Thue-Morse is a list of 01's and 10's. For every 3 digits, there must be one:

? 0 1   0 1 ?   1 0 ?   ? 1 0

It doesn't matter what the ? is, there already can't be a list of 3 equal terms in the Thue-Morse.*

2.11: Because there are no 000 or 111 sequences, by the law in 2.3 there are no 010101 or 101010 sequences either. A 5-digit sequence with no equal consecutive digits requires either one of the invalid sequences:

10101   01010

Because these sequences are impossible, a 5-digit sequence must have a 10 next to a 01 - resulting in a 00 or 11 every single time.

* t_n, t_{n+1}, t_{n+2} is another way of saying "three consecutive t_ns"

*This works for any string to be defined as c and x.

2.12: We need to look at happens when the length of c is >1:

$c\ x\ c\ x\ c$

ab... ab... ab...

Assuming every c can be multiple digits, let's split up these digits:

$(an)\,x\,(an)\,x\,(an)$

Where a is a single binary digit (0 or 1) and where n is all other digits of c. But as long as this configuration is valid (it is defined to be) we can just move the n-digits to the x set of digits, like so:

$a\,(nx)\,a\,(nx)\,a\,n$

Although there is an extra n, the Thue-Morse is infinite, so it is valid to remove it from the "overlap" digits. If we remove the extra n (circled,) replace all a's with c's (it doesn't matter which variable,) and combine all (nx)'s into x's, we find once again:

$c\,x\,c\,x\,c.$

Because of this, we only need c to be a single digit.*

2.13: There are two cases for cxcxc:

- where x is one binary digit (0 or 1),
  If x = c, then we end up with a 000/111 sequence, which is <u>invalid</u>.
  If x ≠ c, then we end up with a 10101/01010 sequence, which is also <u>invalid</u> by the rule from 2.11.
  But there's also the case:
- where x is >1 binary digit (01, 10, ...)
  If x has an even length, it either can be "reversed" (because the Thue-Morse is built on 0->01 and 1->10, we can "reverse" this process and still obtain the Thue-Morse) or has a 000/111, which is <u>invalid</u>.
  The point of "reversing" is to obtain cxcxc where x=1 or x=0, which is already proven to be impossible.
  If x has an odd length >1, x could either be alternating (like 101) or not (like 100 or 110, for example.)
  If x is alternating, such as 101, cxcxc can either be 0<u>101</u>0<u>101</u>0, which breaks the rule from 2.11, or 1<u>101</u>1<u>101</u>1, Which has a 111 and violates 2.10. (cont.)

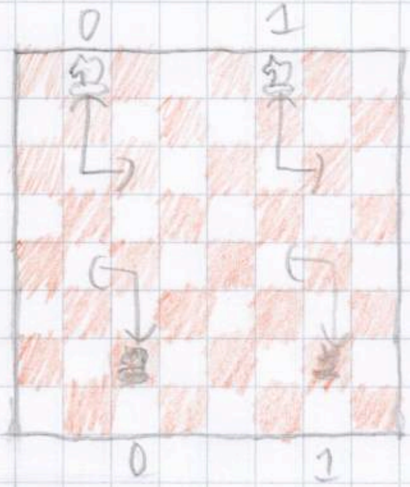
If x does not alternate, such as 100, either 0 100 0 100 0 arises, which contains 000 and is therefore invalid, or 1 100 1 100 1 arises.

However, the Thue-Morse is said to be "square-free," where 1100/0011 in the form of uu never arises. But 110011001 has this property of being "square," therefore it cannot be in the Thue-Morse.

Finally, all sequences with greater odd lengths of x can be created by expanding the invalid ones through the $1 \to 10$ and $0 \to 01$ pattern.

Because all cases have been covered, the Thue-Morse does not have any sequence of the form $cxcxc$. Q.E.D.

2.14: Remember that 2.10 prevents 000 and 111. Because of the 2.3 "expansion law" it also prevents uuu, any three terms of the Thue-Morse repeated in a row - this includes chess moves.



An example of an infinite game is with the knights above infinitely threatening eachother and backing off (This is somewhat realistic, but if you have a lot of time to waste, this is a valid chess move!)

The key part here is that if we assign the left-side knights 0, and 1 likewise for the right, and follow the Thue-Morse, there will <u>never</u> be a pattern of 3 moves.