WORKSHEET 10

The Pigeonhole Principle

PROBLEM 10.1. What is the minimum number of people needed such that you can be *certain* that at least four of them were born on the same day of the week?

PROBLEM 10.2. Show that, among any five points in a square of sidelength 1, there must exist two of distance at most $\frac{1}{\sqrt{2}}$.

PROBLEM 10.3. What is the minimum number of points you need in an equilateral triangle of sidelength 1 such that you can guarantee that there are two points of distance strictly less than $\frac{1}{5}$?

PROBLEM 10.4. Show that among any 51 integers from 1 to 100, we must have at least one pair of consecutive numbers.

PROBLEM 10.5. What is the maximum size of a set of positive integers up to 100 such that no integer in the set is a multiple of another one?

PROBLEM 10.6. Take an 8×8 chessboard, and remove two opposite corner squares. Is it possible to cover the remaining 62 squares using 31 dominoes? (Each domino covers exactly two adjacent squares.)

PROBLEM 10.7. What is the maximum number of knights we can place on an 8×8 chessboard such that no two attack each other?

PROBLEM 10.8. What is the maximum number of bishops we can place on an 8×8 chessboard such that no two attack each other?

PROBLEM 10.9. Show that among $n \geq 2$ people, there must be two people who know the same number of other people. (We assume that knowledge is mutual: if person A knows person B, then person B knows person A.)

PROBLEM 10.10. What is the smallest value of n satisfying the following property: give any n points on the plane such that no three are collinear, there exist four of them forming the vertices of a convex quadrilateral?

Given a sequence of numbers $a_1, a_2, a_3, \ldots, a_k$, a *subsequence* is the sequence formed by taking some collection of the a_i's, in order. For instance, $5, 1, 2$ is a subsequence of $7, 3, 5, 4, 1, 2, 8, 0, 6$.

PROBLEM 10.11. Show that if we have a sequence of $mn+1$ distinct positive integers, we must have an increasing subsequence of length $m+1$ or a decreasing subsequence of length $n+1$.

For instance, suppose $m = 3$ and $n = 2$. In the sequence $4, 1, 6, 7, 3, 5, 2$, the numbers $7, 3, 2$ form a decreasing subsequence of length 3.

The Pigeonhole Principle

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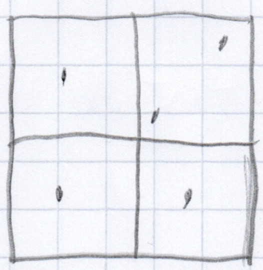
10.1: Consider the worst case scenario:
If everyone kept having birthdays on different days, the week looks like 3 - 3 - 3 - 3 - 3 - 3 - 3.
Sun Mon Tue Wed Thu Fri Sat

Any other person in this week must create a day with 4 birthdays. So the maximum number of people is 3+3+3+3+3+3+3, +1 (to make a 4) = 22.

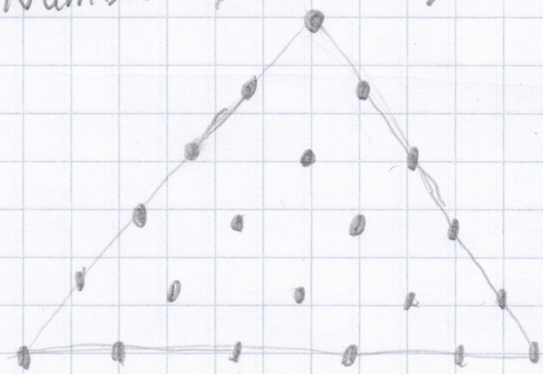
10.2: Divide a square into 4 "holes."



One hole (quadrant) must have 2 dots in it. The maximum length between these 2 dots is just the diagonal of a quadrant/hole or hole, which is $\frac{1}{\sqrt{2}}$ by the Pythagorean Theorem (each side is $\frac{1}{2}$ of big square side length)

*Number symbol again! ‡This forms another valid 50-set.

10.3:



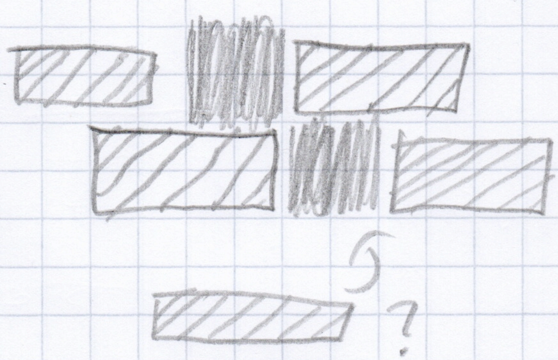
This array shows the maximum possible #* of dots where all are exactly $\frac{1}{5}$ apart; there are 21 of them. So any 22nd dot will satisfy the conditions (it's 22 again!)

10.4: Say we have 50 integers. These can be arranged as: {1, 3, 5, 7 ... 99}. But any 51st ~~number~~ integer must be even, so it will create a consecutive subset of 2 integers. This doesn't work out, so the maximum set size is 50.

10.5: The set of 50 numbers from 51 to 100 satisfies this perfectly. In fact, no matter what the max size is 50: Create sets of #s where each is double the previous: {1, 2, 4, 8...}, {3, 6, 12...}, {5, 10, 20...}... and onward until {99}. No even number can start a set, leaving 50 total sets to choose 50 numbers from!‡

*No multidimensional shenanigans here!

10.6: No! If you remove opposite corners, you remove 2 black squares or 2 white in a chess-board. But every domino must cover both one black and one white square*, so the imbalanced white and dark squares means that there will always be a setup like this in the chessboard:

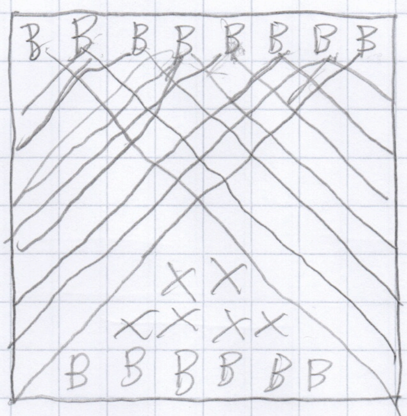


Of course, two squares that are diagonally adjacent like this can't be covered by one domino.

So this tiling will <u>never</u> work.

10.7: Because every knight moves from a light to dark square or vice versa, placing 32 knights on all light or all dark squares ensures that none break their move pattern (light→dark→light) to capture another knight.

10.8:



A row of bishops cannot attack eachother, so let's make row 1 all bishops (B). All of the safe spaces left can be occupied by bishops in multiple ways, but the best (shown) is with 6 bishops. That gives a total of 14 safe bishops on the board.

(Let me know in feedback any other bishop configurations)

*You could also have 1-1-1-1.

10.9: Suppose that $n=2$:

Either 0-0 (nobody knows anyone), or 1-1 (the two each know eachother).

$n=3$:

0-0-0, 1-1-0, 2-2-2 (order ignored).

$n=4$:

0-0-0-0, 1-1-0-0, 2-2-2-0, 3-3-3-3.*

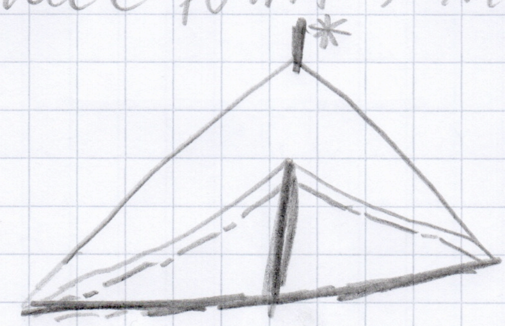
A pattern starts to appear: either everyone is alone, there is a couple, or a trio, or a group of 4... Each person in these "groups" knows $k-1$ people where the group size is $0 \leq k \leq n$. Of course, friendship is mutual, so no single person can know someone that knows 0 people - ensuring that always at least two people know the same number of other people.

*Even expanding the concaveness is collinear.

10.10: Suppose $n=4$:

It's possible to create a concave quadrilateral. But when $n=5$, there are two ways to prevent this from happening:

a) Place point 5 inside this area:



The only way to make a triangle or concave quadrilateral here is for point 5 to be collinear on either two bolded lines - which is impossible.

b). Place point 5 outside of the area. This will always form a convex quadrilateral, just like a)! so $n=5$*.

*These are subsequences.

10.11: Pick any x_i in $\{x_1, x_2, \dots x_{mn+1}\}$. Say that for (a, b), a is the length of the longest increasing sequence* (<u>should</u> be $m+1$) starting with x_i, and that b is the length of the longest decreasing sequence* (<u>should</u> be $n+1$) ending with x_i.

Proof by contradiction: $1 \leq (a, b) \leq (m, n)$ (respectively.) Since there are $mn+1$ x_i terms in the sequence, there are $mn+1$ (a_x, b_y) pairs - but a maximum of mn distinct ones. So, by the Pigeonhole Principle, two x_i terms share the same (a_x, b_y) pair. Call these terms x_r and x_s, where x_r is before x_s. If $x_r < x_s$, then the longest sequence* increasing from x_s would be A, but then x_r would make another $A+1$-length sequence. The same contradiction happens when $x_s < x_r$ on a decreasing sequence, so $(a, b) \geq (m+1, n+1)$