WORKSHEET 20

The Peano Axioms

In this worksheet, we will rigorously construct the set $\mathbb{N}$ of nonnegative integers from axioms. The point is that, while you may be familiar using certain facts about the nonnegative integers, it is also important to know how to prove them from first principles and get used to the axiomatic approach to mathematics, so that in the future when you have to define more exotic objects, you are prepared to deal with the formalism that that entails. When you solve these problems, do not use anything you think you know from elsewhere about the nonnegative integers; you may only use the axioms and previous facts you have deduced from them.

Here are the Peano axioms for constructing the set $\mathbb{N}$ of nonnegative integers:

(1) $\mathbb{N}$ contains an element called 0.
(2) There is a function $S : \mathbb{N} \to \mathbb{N}$, called the *successor function*.
(3) S is injective, i.e. if $S(a) = S(b)$, then $a = b$.
(4) There is no element n in $\mathbb{N}$ such that $S(n) = 0$.
(5) (Principle of Induction.) Suppose A is a subset of $\mathbb{N}$, i.e. a collection of some (possibly none, possibly all) of the elements of $\mathbb{N}$, such that 0 is in A, and whenever n is in A, then $S(n)$ is in A as well. Then $A = \mathbb{N}$.

The successor function S is the axiomatic version of the "add one" function, i.e. we should think of $S(n)$ as $n + 1$.

PROBLEM 20.1. Show that for all n in $\mathbb{N}$ other than $n = 0$, there exists some a in $\mathbb{N}$ such that $n = S(a)$. (You will need to use the Principle of Induction.)

We next define addition, again directly in terms of the axioms.Next, we define addition by 1: for all a in $\mathbb{N}$, we define $a + 0$ to be a. Next, if a and b are in $\mathbb{N}$, then we define

$$a + S(b) = S(a + b).$$

PROBLEM 20.2. Define $1 = S(0)$ and $2 = S(1)$. Verify that you can express $a + 1$ and $a + 2$ in terms of the successor function, directly from the definition of addition.

PROBLEM 20.3. Show that if a and b are in $\mathbb{N}$, then $a+b=b+a$. (You will need to use the Principle of Induction for this. The way to set this up is to define a set $B = \{b : a+b = b+a \text{ for all } a \text{ in } \mathbb{N}\}$. This is a typical trick that you will need to use many times in the remaining problems.)

PROBLEM 20.4. Show that if a, b, c are in $\mathbb{N}$, then $a + (b + c) = (a + b) + c$.

Next, we define multiplication. As with addition, we start with multiplication by 0: if a is in $\mathbb{N}$, we define $a \cdot 0 = 0$. Next, if a and b are in $\mathbb{N}$, we define

$$a \cdot S(b) = a \cdot b + a.$$

PROBLEM 20.5. Verify that you can express $a \cdot 1$ and $a \cdot 2$ in terms of addition, directly from the definition of multiplication.

PROBLEM 20.6. Show that if a, b, c are in $\mathbb{N}$ and $a+c=b+c$, then $a=b$.

PROBLEM 20.7. Show that if a, b, c are in $\mathbb{N}$, then $a \cdot (b + c) = a \cdot b + a \cdot c$.

PROBLEM 20.8. Show that if a, b, c are in $\mathbb{N}$, then $a(bc) = (ab)c$.

PROBLEM 20.9. Show that if a and b are in $\mathbb{N}$, then $ab = ba$.

We also need to talk about the ordering on $\mathbb{N}$. If a and b are in $\mathbb{N}$, we say that $a \leq b$ if there is some c in $\mathbb{N}$ such that $a+c=b$. We say that $a < b$ if $a \leq b$ and $b \not\leq a$.

PROBLEM 20.10. Show that if $a \leq b$ and $b \leq c$, then $a \leq c$.

PROBLEM 20.11. Show that if a is in $\mathbb{N}$, then $a \neq a+1$. More generally, show that if a and k are in $\mathbb{N}$ and $k \neq 0$, then $a + k \neq a$.

PROBLEM 20.12. Show that if a and b are in $\mathbb{N}$, then exactly one of the following is true:

- $a < b$,
- $a = b$,
- $b < a$.

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20.1: Suppose that there exists a set A such that $a \in A$. By Axiom 5, $S(a) = n$ is also in A and $A = \mathbb{N}$. Therefore, an a such that $S(a) = n$ must exist in $\mathbb{N}$.

20.2: Yes - $a+1 = a+S(0)$, and $a+2 = a+S(1) = a+S(S(0))$.
Because $x + S(y) = S(x+y)$, $a+S(0) = S(a)$ and $a+S(1) = S(S(a))$.

20.3: Say that, for the set B, B = {all such b which satisfy $a+b=b+a$ with all a in n.}
Say that B is not empty, because $b=0$ inherently must be in B ($a+0=0+a$).
If $S(b)$ is in B, then by induction $B = \mathbb{N}$.
So, $B = \{b : a+S(b) = S(b)+a, a \in \mathbb{N}.\}$
$= S(a+b) = S(b+a), a \in \mathbb{N}.$
$a+b = b+a$, so $S(b)$ is also in B, and by (5) $B = \mathbb{N}$.

*This creates 20.3's case. ‡Times 1 higher y = 1 more x than $x \cdot y$

20.4: Same thing: $B = \{b, c : a+(b+c) = (a+b)+c, a \in \mathbb{N}.\}$

c can be 0*, so B is not empty.

If B includes $S(c)$, then by (5) $B = \mathbb{N}$.

So: $B = \{a+(b+S(c)) = (a+b)+S(c)\}$

$= \{a+S(b+c) = S((a+b)+c)\}$

$= \{S(a+(b+c)) = S((a+b)+c)\}$.

$a+(b+c) = (a+b)+c$, so $S(c)$ is also in B, and by (5) $B = \mathbb{N}$.

20.5: Yes - $a \cdot 1 = a \cdot S(0)$, and

$a \cdot 2 = a \cdot S(1) = a \cdot S(S(0))$.

Because $x \cdot S(y) = x + (x \cdot y)$‡,

$a \cdot S(0) = a + (a \cdot 0) = a$, and

$a \cdot S(1) = a + (a \cdot 1) = a + a$.

20.6: Again, set $B = \{a, b : a+c = b+c$ means $a = b$ for all $c \in \mathbb{N}.\}$

c can be 0 where $a = b$, so B is not empty.

If B has $S(c)$ then by (5) $B = \mathbb{N}$.

So: $B = \{a+S(c) = b+S(c) = S(a+c) = S(b+c).\}$

By Axiom (3), S is injective, so $S(a+c) = S(b+c)$ means that $a+c$ still equals $b+c$.

By Axiom (5) and because $S(c)$ is in B, then $B = \mathbb{N}$ and all $\mathbb{N}$ satisfy this.

*This works by the rule from 20.5 other than $a \cdot 0 = 0$.

20.7: Set $B = \{b, c : a \cdot (b+c) = a \cdot b + a \cdot c \text{ for all } a \in \mathbb{N}.\}$

$a = 0$ works, so B is not empty.

If B has $S(a)$, then by (5) $B = \mathbb{N}$.

So: $B = \{S(a) \cdot (b+c) = S(a) \cdot b + S(a) \cdot c\}$

$= b + c + ab + ac = b + ab + c + ac$*

Of course, these expressions are equivalent, so $S(a) \in B$.

By Axiom (5), therefore $B = \mathbb{N}$.

20.8: Set $B = \{b, c : a(bc) = (ab)c \text{ for all } a \in \mathbb{N}.\}$

$a = 0$ works, so B is not empty.

If B has $S(a)$, then by (5) $B = \mathbb{N}$.

So: $B = \{S(a) \cdot (bc) = (S(a)b) \cdot c\}$

$= bc + abc = (b + ab) \cdot c$.

By 20.7's proof, $(b + ab) \cdot c = bc + abc$.

Therefore, these expressions are equal, and by (5) and because $S(a) \in \mathbb{N}$, $B = \mathbb{N}$.

20.9: Set $B = \{b : ab = ba \text{ for all } a \in \mathbb{N}.\}$ $a = 0$ works, so B is not empty. Testing $S(a)$:

$B = \{b : S(a) \cdot b = b \cdot S(a) = b + ab = b + ab\}$.

These expressions equal shows that $S(a) \in B$, so $B = \mathbb{N}$.

*$k \neq 0$.

20.10: If $a \leq b$ and $b \leq c$ then by definition there exists n, k such that $a+n=b$ and $b+k=c$. If $a \leq c$, then a value exists where $a+?=c$. Substitute: $b+k=c=a+n+k=c$. $(n+k)$ is a value like the ?, and therefore $a \leq c$.

20.11: set $B=\{b: b \neq b+1$, for all $b \in \mathbb{N}\}$. $1 \in B$, because otherwise $1=S(1)$, and therefore $0=S(0)$ which is impossible by Axiom (4). If $S(b)$ is in B, then by (5) $B=\mathbb{N}$. So: $B=\{S(b) \neq S(b)+1=S(b)+S(0)=S(S(b))\}$. Of course, by Axiom 3, $S(b) \neq S(S(b))$ rewrites to $b \neq S(b)=b+1$. This means that $b \neq b+1$ holds true for $S(b)$, and therefore by axiom (5) $B=\mathbb{N}$. General: $B=\{b: b \neq b+k$, for all $b \in \mathbb{N}.\}$ If $S(k) \in B$, then by (5) $B=\mathbb{N}$. So: $B=\{b \neq b+S(k)\}$*...

... $=S(b+k)$. Now, if $S(b)$ is in B, this holds.

Assume the contradiction... (cont.)

*Technically a different B. ‡This could be $S(a)=b$, which also is Case 2.

20.11: (cont.): in which $b = S(b)+S(k)$.

$S(b)+S(k) = S(b+S(k)) = S(S(b+k))$.

Injectivity (3) says that $b = S(b+k)$.

However, $b = S(b+k) = b+S(k) \neq S(b)+S(k)$.

Therefore, $S(b)$ does not also satisfy $b=b+k$, and set $B^* = \{b : b=b+k$ for all $k, b \in \mathbb{N}\}$ cannot equal $\mathbb{N}$, so $b \neq b+k$.

20.12: For either $a<b$, $a=b$, or $a>b$ to be true, each must work with $S(a)$.

Case 1: $S(a)<b$‡. Here, for some $k \in \mathbb{N}$, $S(a)+k=b$. $S(a+k)=b$. If $k=0$, then $S(a)=b$‡ and $a<b$. If $k \neq 0$, $S(a) \neq b$. If this means that $S(a)+k=b$, then $S(a)<b$ is true. Suppose that $b=a+S(k)$, which must be true because $a<b$. This means that $b=a+S(k)=S(a+k)=S(a)+k$, so $S(a)<b$ is also true (unless $k=0$, where $S(a)=b$.)

Case 2: $a=b$ and $S(a)>b$. If $a=b$, then $S(a)=S(b)$ by (3). $S(a)=S(b)=S(b+0)=b+S(0)$. If $k=S(0)$, then $S(a)=b+k$, so $S(a)>b$. If $S(a)=b=b+S(0)$, then $0=S(0)$ which is impossible (4).

*If $a>b$ then $S(a)>S(b)$ by injectivity.

20.12 (cont.): Case 3: $a>b$ and $S(a)>b$.

This would mean that $S(a)=S(b)+k$*...

... $=S(b+k)=S(k)+b$. $k\neq 0$, because otherwise $b=b+S(k)$ and $S(k)=0$, which is impossible by (4). So $S(a)\neq b$, and $S(a)>b$ if $a>b$.

Finally, show that only one ($a<b$, $a=b$, $a>b$) is true at any time. $a<b$ or $a>b$ means that $a\neq b$, so if $a=b$ then $a<b$ or $a>b$ is false. What if $a<b$ and $a>b$? Then we can write $a=b+k$ and $b=a+n$. Substitute: $a=a+n+k$, so $0=n+k$. No number following the axioms can add to 0 (4), so $a<b$ and $a>b$ cannot both be true.

Now that we have proven that $a<b$, $a=b$, and $a>b$ are in $\mathbb{N}$, but no two (or three) can be true at once, this means that only one of $a<b$, $a=b$, or $a>b$ is true.