WORKSHEET 4

Symmetries

Consider a square with vertices at $(\pm 1, \pm 1)$. If we reflect it about the x-axis, we get the same square again. Similarly, if we rotate it by $90°$ counterclockwise, we get the same square again. However, if we rotate it by $45°$, we get a different square. See Figure 1.

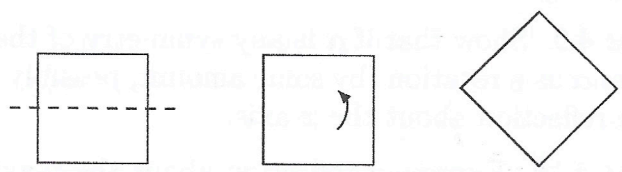

Figure 1. If we reflect the square across the horizontal line (Left) or rotate it $90°$ counterclockwise (Center), we get the same square, whereas if we rotate it by $45°$ counterclockwise (Right), we get a different square.

We call a rotation or reflection of the square a *symmetry* of the square if, after performing it, we get the same square we started with. We include the "identity" symmetry, that does nothing. Each symmetry of the square can be considered as a function. For instance, the reflection about the x-axis can be described as $f(x, y) = (x, -y)$. We consider two symmetries to be the same if they are the same function. For instance, if we reflect about the x-axis twice, then we get back to the identity function $f(x, y) = (x, y)$, so it is considered to be the same as the identity function.

PROBLEM 4.1. How many symmetries are there of the square?

PROBLEM 4.2. More generally, how many symmetries are there of a regular n-gon?

PROBLEM 4.3. Find a polygon with exactly 3 symmetries. More generally, for every positive integer n, find a polygon with exactly n symmetries.

PROBLEM 4.4. Let P be a polygon. Show that if there is a line ℓ such that reflection about ℓ is a symmetry of P, then P has an even number of symmetries.

PROBLEM 4.5. Find a polygon that isn't a square, but that has exactly the same symmetries as a square.

→ 16-gon w/ the pointy Δs

PROBLEM 4.6. We can extend the notion of symmetries to 3-dimensional polyhedra. How many symmetries does a cube have?

→ 

PROBLEM 4.7. Let D_4 be the set of symmetries of the square.[1] Let τ be the reflection about the x-axis, and let ρ be the rotation by 90° counterclockwise. What is $\tau \circ \rho \circ \tau$? That is, what happens when you perform τ, then ρ, then τ?

PROBLEM 4.8. Suppose α and β are two symmetries of a polygon P. Show that $\alpha \circ \beta$ is another symmetry of P. (Note that $\alpha \circ \beta$ means that you perform β first, then α. For instance if α is a rotation and β is a reflection, then $\alpha \circ \beta$ means that you perform the reflection first, then the rotation.)

PROBLEM 4.9. Show that if α is any symmetry of the square, then we can express α as a rotation (by some amount, possibly 0), optionally followed by a reflection about the x-axis.

PROBLEM 4.10. Express a reflection about the y-axis in terms of rotations and perhaps a reflection about the x-axis, as in Problem 4.9.

PROBLEM 4.11. Express a reflection about the y-axis followed by a reflection about the line $y = x$ as in Problem 4.9.

PROBLEM 4.12. Generalize Problem 4.9 to symmetries of a regular n-gon.

Extending the notation from Problem 4.7, let ρ^{-1} be a rotation by 90° clockwise. Say that two elements α and β of D_4 are *directly conjugate* if $\alpha = \tau \circ \beta \circ \tau$ or $\alpha = \rho \circ \beta \circ \rho^{-1}$, or vice versa. Say that α and β are *conjugate* if there exist elements $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_k = \beta$ such that for each i with $1 \leq i \leq k-1$, the elements α_i and α_{i+1} are directly conjugate. Write $\alpha \sim \beta$ if α and β are conjugate.

PROBLEM 4.13. Show that if α is in D_4, then $\alpha \sim \alpha$.

→ There is no (a_i, a_{i+1}) to exist, so it must work.

PROBLEM 4.14. Show that if α and β are in D_4, and $\alpha \sim \beta$, then $\beta \sim \alpha$.

If you can create one symmetry from another, you must be able to conjugately vice versa.

PROBLEM 4.15. Show that if α, β, and γ are in D_4, and $\alpha \sim \beta$ and $\beta \sim \gamma$, then $\alpha \sim \gamma$.

→ This is saying that if $\beta \sim \alpha$, than $\alpha \sim \gamma$ is a must do.

It follows from the last three problems that the elements of D_4 split up into *conjugacy classes*. Given α in D_4, write $[\alpha]$ for the set of all β in D_4 such that $\alpha \sim \beta$, including α itself.

PROBLEM 4.16. Find all the elements in $[\rho]$, $[\rho \circ \rho]$, and $[\tau]$.

[1]The D stands for "dihedral."

$[\rho] = 90°, 270° = \rho, \rho^3$. cannot get more.

$[\rho\circ\rho] = 180° = \rho^2$. Also can't get more (only rotating)

$[\tau] = \tau, \rho\tau\rho^{-1}, \rho^3\tau\rho^{-3}$.

Symmetries

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4.1: The square has eight symmetries:

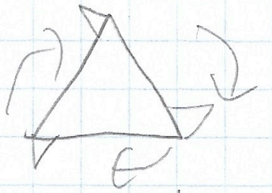
Reflections, □, □, □, □.

Rotations, □ 90°, □ Flip 180°, □.

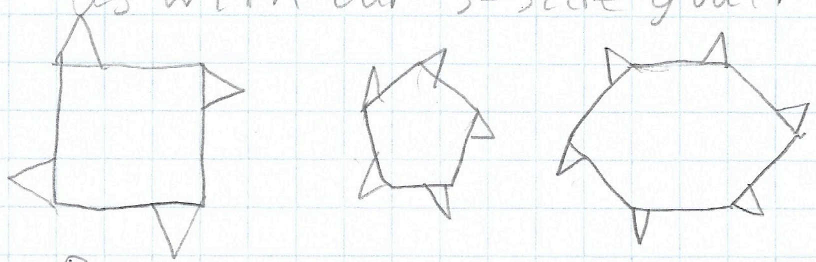
Identity, □. (This is also a 360° rotation.)

4.2: Generalizing, every side has a symmetry and every center-crossing diagonal is also a symmetry. But every side repeats this with the opposite side, so dividing by 2 gains the number of sides / the number of center-crossing diagonals. Also, every side has 1 rotation from a particular side (and the 360° one is identity.) Adding it all up, the symmetries of a regular n-gon sum to a list of 2n total symmetries.

4.3: It turns out that the closest polygons we are looking for is an equilateral triangle, with 3 rotations and 3 reflections. But if we do this:



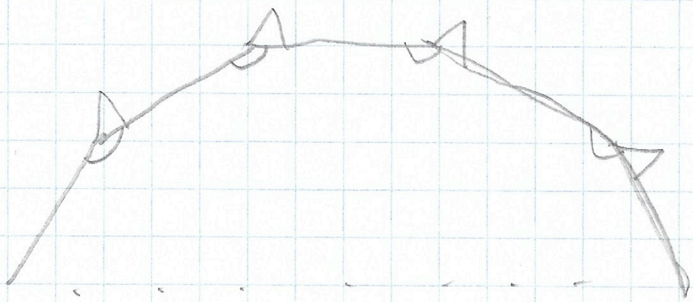
rotational symmetry still works, (counting 360° as the identity) but all reflections fail, leaving us with our 3-side goal.



But this same strategy also works for any n-gon, leaving n rotational symmetries!

4.4: If we know that the line ℓ creates a symmetry of the chosen polygon, then we can pair it with the 360°/identity symmetry. Any other reflection symmetries must be accompanied by a rotation symmetry just like the original reflection on line ℓ. Therefore the only possible symmetries are 2, 4, 6, 8... basically even numbers.

4.5:



This is just a small piece, but another 8-symmetry shape is simply a 16-gon with all reflective symmetries (half) removed using the Δ's. $\frac{16}{2}$ is 8, so we're done!

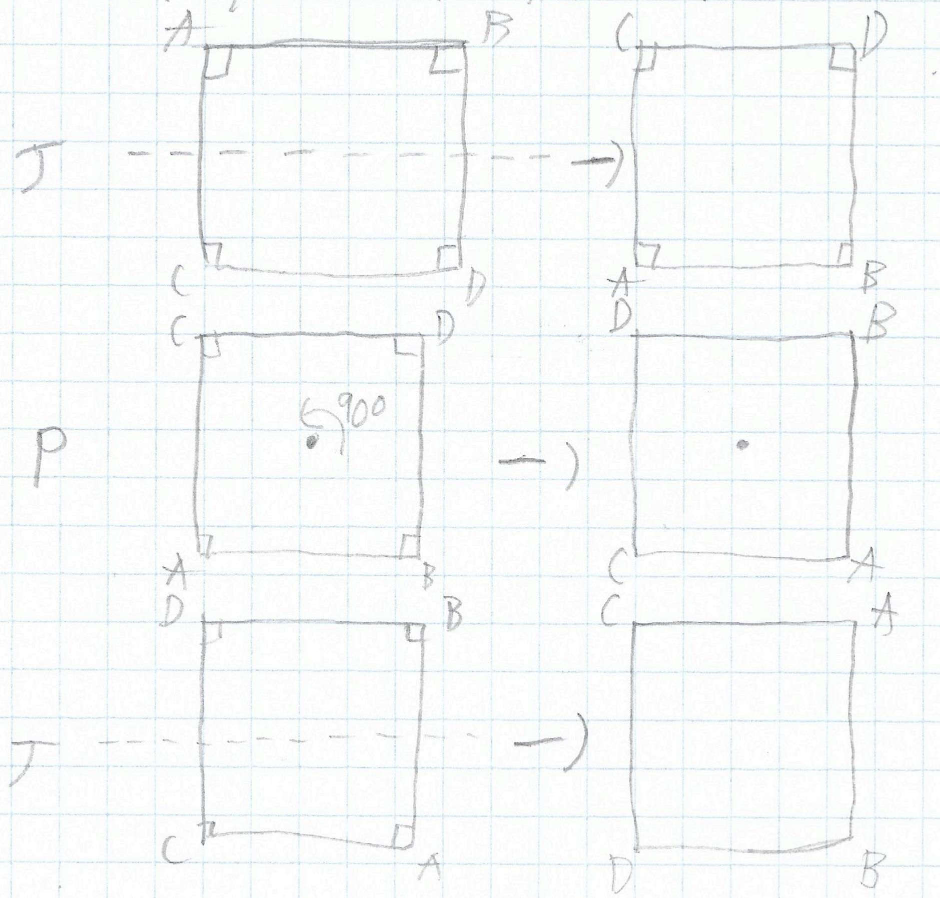
4.6: The two types of cube symmetries:

Rotational (axes)	Reflection (planes)
• 3 center-face axes, 9 ways total	• 3 center-opposite-face planes
• 4 diagonal opposite-vertice axes, 8 ways total	• 6 diagonal edge planes
• 6 center-opposite-edge axes, 6 ways total	• Note: The cube has more two-step reflect + rotate symmetries that aren't counted for simplicity.

(cont.)

4.6(cont.): Adding this up, there are a total of $24+9=\underline{33}$ cube symmetries (in one step. there are 48 for two-step included.)

4.7: Let's try this with □ABCD:



Note how the end result is simply a 90° rotation of □ABCD, likely because symmetries are communative and both J reflections cancelled out. (The square itself doesn't change, but the vertices do.)

4.8: Because a symmetry is an operation that when performed outputs the same shape, two symmetries α and $\beta = \alpha \circ \beta = x$. Because both α and β result in the same shape, x also results in the same shape and is therefore a symmetry.

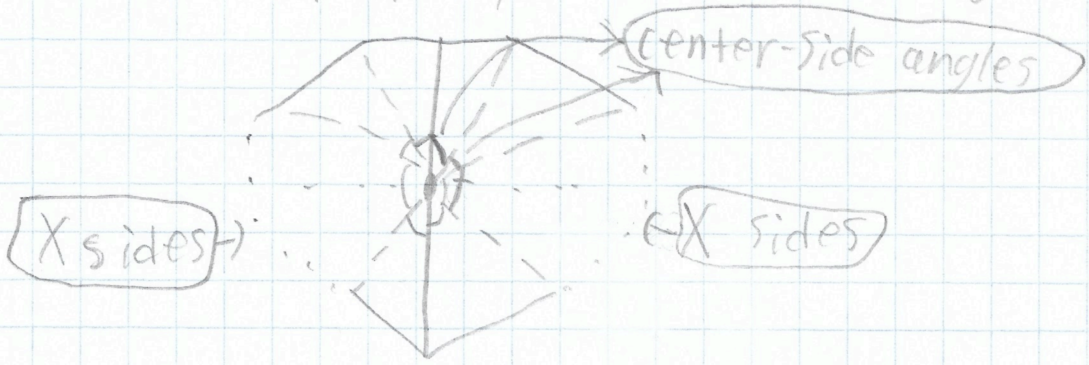
4.9: It doesn't matter what α is - we can express it as a rotational symmetry because both α and another rotational symmetry result in the same square. From the 4.8 rule, this also works for more complicated symmetries.

4.10: This is essentially the case of 4.9 that is a reflection about the y-axis for α, meaning that α can be expressed as a rotation (ignoring vertices) as well as a two-step symmetry like in 4.8.

*or reflection! †or an axis of many

4.11: Here, because both $y=x$ and the y-axis are symmetries of the square, any α rotation* will do (Again, ignoring the vertices) as an expression.

4.12: Another property shows 4.9 for n-gons:



If line t is the† axis of symmetry for this n-gon, any rotation by the center angles proportional to each side will create another axis of symmetry. Therefore, because each rotation by a center-side angle is a symmetry in itself, each axis can be expressed as the rotation that creates it from one axis of symmetry, α. (If α is not a reflection, then it is already a rotation!)

4.13: Because there is no (a_i, a_{i+1}) elements in a conjugacy set of size 1 $(a \sim a_1)$ α must be directly conjugate to itself and is also, therefore, conjugate.

4.14: If we look at the conjugacy definition:

$$\alpha = a_1, a_2, \ldots, a_K = \beta,$$

then it is valid to rearrange this as:

$$\beta = a_K, a_{K-1}, \ldots, a_2, a_1 = a.$$

In simpler terms, performing a number of symmetrical transformations to get from the element α to β can also be expressed in reverse, where each symmetry a's are performed starting with a_K down to a_1 instead of vice versa.

4.15: If we say that $\beta = a_n$ and that $\gamma = \beta$ (previously) then we get the standard conjugacy relationship all over again, meaning that $a \sim \gamma$.

4.16:

- set $[\rho]$:
  - $\rho \rho \rho^{-1} = \rho$, ρ itself.
  - $\tau \rho \tau = \rho^3$, 90° clockwise because directions flip.

$\Rightarrow$ $[\rho]$ has two elements, $\tau \circ \rho \circ \tau = \rho^3$ and just ρ.

- set $[\rho \circ \rho]$ (basically ρ^2)
  - Every operation just resuclts in ρ^2.

$\Rightarrow$ $[\rho \circ \rho]$ has one element, ρ^2.

- set $[\tau]$ (x-axis)
  - $\rho \tau \rho^{-1}$ = reflection across y-axis (call it $\tau \rho^2$)
  - $\tau \tau \tau = \tau$, no change

$\Rightarrow$ $[\tau]$ has two elements, $\tau \rho^2$ and τ.

Note: Diagonal reflections in D_4 do not appear here because they aren't conjugately reachable by just ρ and τ. There are many more symmetries of the square (and n-gons) not explored here.