WORKSHEET 11

Permutations

A *permutation* of n is an ordering of the numbers from 1 to n. For instance, 6253174 is a permutation of 7.

PROBLEM 11.1. How many permutations are there of n?

We write S_n for the set of all permutations of n.

Given any permutation, we can reach it starting from the "identity" permutation $123\cdots n$ by doing a bunch of swaps. For instance, for the permutation above, we can reach it by doing the following swaps (marked in red):

1234567

5234167

5243167

5273164

7253164

6253174

At the end, we reached the desired permutation.

PROBLEM 11.2. In the above example, not all the swaps were of numbers in consecutive positions. Show that it is possible to reach any permutation by means of a sequence of consecutive swaps.

Suppose τ is a permutation of n. For $1 \leq i \leq n-1$, let $\sigma_i(\tau)$ be the permutation obtained from τ by swapping the numbers in the i^{th} and $(i+1)^{\text{st}}$ positions in τ. For instance, we have

$$\sigma_3(5146372) = 5164372.$$

PROBLEM 11.3. Show that for any permutation τ of n and any i with $1 \leq i \leq n-1$, we have $\sigma_i(\sigma_i(\tau)) = \tau$.

PROBLEM 11.4. Show that if τ is a permutation of n and $1 \leq i, j \leq n-1$ with $|i-j| \geq 2$, then $\sigma_i(\sigma_j(\tau)) = \sigma_j(\sigma_i(\tau))$. Show that this is not necessarily true if $|i-j| = 1$.

PROBLEM 11.5. Show that if τ is a permutation of n and $1 \leq i \leq n-2$, then $\sigma_i(\sigma_{i+1}(\sigma_i(\tau))) = \sigma_{i+1}(\sigma_i(\sigma_{i+1}(\tau)))$. abc abc

1 2 1

bac acb

bca cab

cba cba

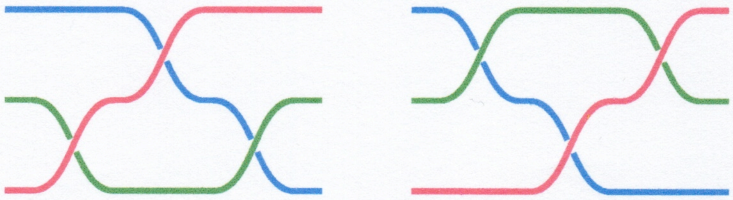


Figure 1. A graphical representation of the identity $\sigma_i \circ \sigma_{i+1} \circ \sigma_i = \sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1}$.

1234567
2134567
2314567
2341567
2345167
2345617
2345671
2345617

We can draw a picture of this; see Figure 1.

PROBLEM 11.6. For any swap ρ of two numbers in a permutation, we can undo the swap by doing it again; that is, $\rho(\rho(\tau)) = \tau$. But what happens if we do a sequence of swaps, say $\rho_1 \circ \rho_2 \circ \cdots \circ \rho_k$, where each ρ_i is a swap? What do we have to do to undo this entire sequence of swaps?

PROBLEM 11.7. Suppose that $\rho_1, \ldots, \rho_k$ and $\pi_1, \ldots, \pi_m$ are swaps such that $\rho_1 \circ \cdots \circ \rho_k(\tau) = \pi_1 \circ \cdots \circ \pi_m(\tau)$. Show that $\rho_1 \circ \cdots \circ \rho_k \circ \pi_m \circ \pi_{m-1} \circ \cdots \circ \pi_1(\tau) = \tau$.

In the preceding problems, there's no meaningful dependence on τ, since the same identities hold for all τ. Thus we'll write things like $\rho_1 \circ \cdots \circ \rho_k = \pi_1 \circ \cdots \circ \pi_m$.

PROBLEM 11.8. Explain how to write an arbitrary swap in terms of a sequence of consecutive swaps.

PROBLEM 11.9. Suppose that we have two sequences of swaps $\rho_1, \ldots, \rho_k$ and $\pi_1, \ldots, \pi_m$ such that $\rho_1 \circ \cdots \circ \rho_k = \pi_1 \circ \cdots \circ \pi_m$. Show that if k is even, then m is as well, and vice versa.

We write A_n for the set of permutations that are obtainable from the identity permutation by means of an even number of swaps in some (hence any) way. The set A_n is called the *alternating group*.

PROBLEM 11.10. How many elements does A_n have?

There is another very important way of representing permutations. Given a permutation τ written out as a sequence of numbers, we can consider it as a function $\tau : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ by letting the first number denote $\tau(1)$, the second number $\tau(2)$, and so on. For instance, the permutation 6253174 is the function τ such that $\tau(1) = 6$, $\tau(2) = 2$, $\tau(3) = 5$, $\tau(4) = 3$, $\tau(5) = 1$, $\tau(6) = 7$, and $\tau(7) = 4$.

It is natural to represent a function by means of a picture, where for each i, we draw an arrow from i to $\tau(i)$. The picture for this permutation τ is shown in Figure 2.

We can represent this picture more succinctly, by writing each cycle in parentheses in order. The permutation 6253174 is then (167435)(2).

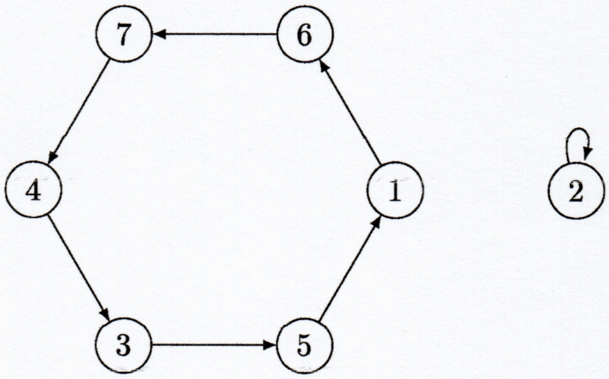


Figure 2. A picture of the permutation 6253174.

PROBLEM 11.11. If you are given the cycle decomposition of a permutation, how can you tell quickly whether or not it lies in the alternating group?

Permutations

Alexander F. 1/13-18/26

11.1: There are n slots to place any number from 1 to n once in each. This is modeled exactly by n! (factorial.)

11.2: Think of this like a one-dimensional slider puzzle:

$\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}\boxed{6}\boxed{7}$ → 6253174

Swapping only 2 consecutive numbers:
1234657 → 1236457 → 1263457 → 1623457 → 6123457. The 6 is in the "correct spot," so it can be ignored. 123457 → 213457. Now ignore the 2: 13457 → 13547 → 15347 → 51347. Ignore the 5: 1347 → 3147. Next, 147 → 174.

Now we're done! 6253174.

Any permutation can change into any other of the same length with only consecutive "slider puzzle" swaps.

11.3: Example: abcdef. ($=\tau$)

$\sigma_3(\tau)$=abdcef. and $\sigma_3(\sigma_3(\tau))$=abcdef again. Therefore, σ_i is an "involution" (dictionary.)

11.4: As long as the difference between σ_i and σ_j is greater than 2 (i.e. the functions do not overlap.) But if the difference is 1, the swaps affect the same area, so order does matter where the center number, being affected by both swaps, goes to.

11.5: These σ functions only affect 3 elements in the permutation- let's call them a, b, and c.

Function 1: abc -) bac -) bca -) cba

Function 2: abc -) acb -) cab -) cba

They return the same result no matter the specific element!

11.6: This sequence ends up moving the first term to the last, for example 1234567 → 2345671. So if we perform $p_k \circ \dots \circ p_1$) or the reverse order, this carries the last term back to the first - the original sequence.

11.7: We already know from 11.6 that the inverse of $p_1 \circ \dots \circ p_k$ is $p_k \circ \dots \circ p_1$. So if another function has the same exact output, its inverse is also $p_k \circ \dots \circ p_1$. This other function's inverse must cancel out with $p_1 \circ \dots \circ p_k$ too, so the equation works.

11.8: The trick here is to use swaps to put the two target terms next to each other, perform the swap, and then return the swapped term back to its... (cont.)

11.8(cont.): ... original position. Ex:

1234567 = 1234567 → 2134567

→ 2314567 → 2341567 → 2345167

→ 2354167 → 2534167 → 5234167.

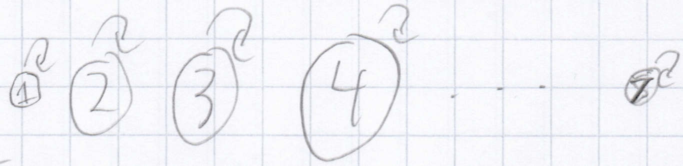
Swap complete with only consecutive swaps.

11.9: Suppose that m parity ≠ k parity. In this case, each sequence of swaps must line up perfectly, but an odd number of swaps will always have one more or less swap than an even number of swaps, which will create a different permutation. So m parity = k parity, and if m is even, then k must be even.

11.10: Each odd permutation can be changed into an even one with one swap. Therefore, each even permutation can be paired with each odd one, so the even permutations are $\frac{n!}{2}$ (n! total.)

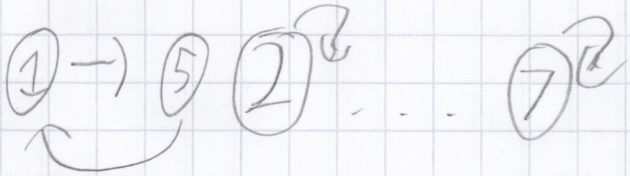
* implying a permutation

11.11: The cycle decomposition (CD) of 1234567:



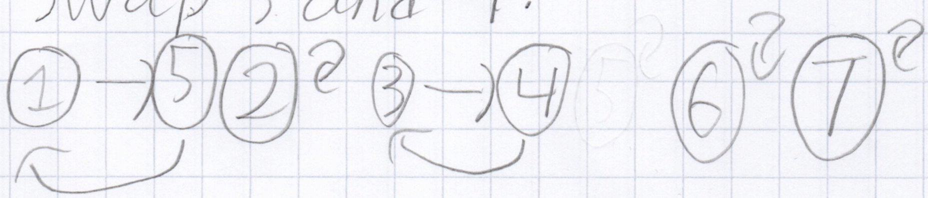
Each number represents itself.

Swap 5 and 1 (for example):



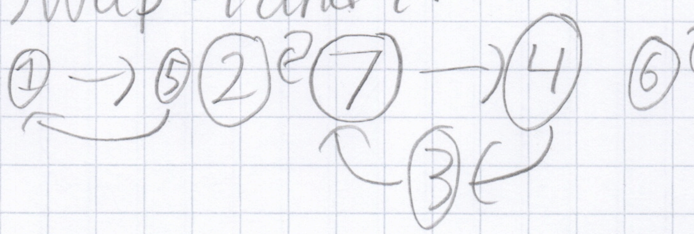
This odd sequence* has 1 cycle of 2.

Swap 3 and 4:



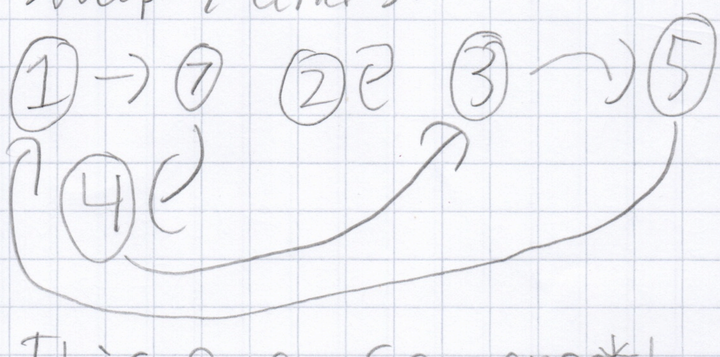
This even sequence* has 2 cycles of 2.

Swap 4 and 7:



This odd sequence* has 2 cycles of 2/3.

Swap 7 and 5:



This even sequence* has 1 cycle of 5! . . .

11.11(cont.)... The pattern seems to show that each CD has x cycles of lengths y, z, etc.; and for each CD with lengths of cycles ≥ 2, counting those ≥ 2 cycles only, this pattern emerges:

- Odd permutations: x+y+z+etc. = Odd
- Even: x+y+z+etc. = Even

And there you have it: an efficient way to find parity of permutations from their CD.

11.12: Don't read this solution unless you really want to!