WORKSHEET 12

Permutation Statistics

0-123-3
1-132-1
1-213-1
0-231-0
0-312-0
1-321-1
=6

Let τ be a permutation of n, thought of as a function from $\{1, 2, \ldots, n\}$ to itself. A number i with $1 \leq i \leq n$ is said to be a *fixed point* of τ if $\tau(i) = i$.

PROBLEM 12.1. How many permutations of n are there such that 1 is a fixed point? Such that 2 is a fixed point? Generalize.

1234-4
2134-2-1
2314-1
2341-0
2143-0-2
2413-0
2431-1
1243-2-1
1324-2-1
1423-1
1342-1
1432-2-1
3124-1
3142-0
3241-1
3421-0
3214-2-1
3412-0-2
4123-0
4132-1
4213-1
4312-0
4231-2-1
4321-0-2
=24

PROBLEM 12.2. What is the total number of fixed points among all permutations of n? What is the average number of fixed points?

In the cycle decomposition of τ, τ splits up into some number of cycles of length 1, i.e. fixed points, some number of cycles of length 2, some number of cycles of length 3, and so on. From the previous problem, we now know the average number of fixed points across all permutations. Now we would like to know the averages of other cycle lengths as well.

PROBLEM 12.3. How many permutations τ of n are there such that (12) is a cycle of length 2 in τ? What about (13)? Generalize.

PROBLEM 12.4. What is the average number of cycles of length 2 among all permutations of n?

PROBLEM 12.5. What is the average number of cycles of length k among all permutations of n?

PROBLEM 12.6. What is the average number of cycles (of any length) among all permutations of n?

Hint: sum all lengths, giving $(1+\frac{1}{2}+\frac{1}{3}\cdots+\frac{1}{n})$

PROBLEM 12.7. Show that a permutation of n cannot have more than one cycle of length $> \frac{n}{2}$, and more generally that it cannot have more than k cycles of length $> \frac{n}{k+1}$.

Here is a very neat and surprising application of our results so far on permutation statistics:

A warden in charge of 100 prisoners plays the following game with the prisoners. There are 100 boxes labeled 1–100 and 100 pieces of paper, each containing a distinct number from 1–100. The warden randomly places the papers into the boxes, with one paper going into each box. Each prisoner is also assigned a distinct number from 1–100.

One by one, each prisoner is brought into the room with the boxes and is allowed to open 50 boxes. If all the prisoners manage to find their own number among the 50 boxes they open, then they are all free, but if at least one prisoner fails to find eir[1] number, then all the prisoners have to remain in prison.

PROBLEM 12.8. Suppose each prisoner chooses 50 boxes randomly. What is the probability that they go free?

PROBLEM 12.9. Clearly, random choice is not a good strategy for success. Come up with a better strategy for them to go free. How well can you do? What is the approximate probability of freedom? (You may wish to use the fact that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \approx \log k + \gamma$, where log is the natural logarithm (to base e), and $\gamma \approx 0.57721$ is a number.)

PROBLEM 12.10. 100 passengers are about to board an airplane with 100 seats. Everyone has a ticket with an assigned seat. However, the first passenger has lost eir ticket and sits in a random seat. Every subsequent passenger attempts to take eir assigned seat, but if it is already taken, then ey chooses a random unoccupied seat. You are the last passenger in line to board, so there is only one seat left when you get on the plane. What is the probability that you get to sit in your assigned seat?

PROBLEM 12.11. If you choose a random permutation of n, what is the probability that 1 and 2 are in the same cycle?

PROBLEM 12.12. If you choose a random permutation of n, what is the probability that 1, 2, and 3 are in the same cycle?

[1]This is an example of a *Spivak pronoun*, which is a third-person singular gender-neutral pronoun. To form these pronouns, take the third-person plural pronouns, which start with "th," and then remove the "th," producing "ey," "em," "eir," and so forth. The only exception is "themselves," which becomes "emself." One of my missions is to make Spivak pronouns into a commonly used part of the English language. Please do your part to support the cause!

Permutation Stats.

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12.1: If $\tau(1)=1$, then you have $n-1$ choices for what $\tau(2)$ equals, $n-2$ choices for $\tau(3)$, and so on until you have only 1 choice for $\tau(n.)$ The total number of such permutations is $(n-1)\cdot(n-2)\cdot\ldots\cdot2\cdot1$. In other words, $(n-1)!$. This also works for any other fixed point - the thought process is similar.

12.2: It turns out that the total number of fixed points is always equal to the total number of permutations for 1-n. So (1) has 1, (12) has 2, (123) has 6, (1234) has 24, and so on. The avg is just $\frac{\text{total fixed pts.}}{\text{total permutations}}$, Which always equals 1.

*This is also $(n-2)!$ permutations.

12.3: If you swap any two values, each being fixed, then there are $n-2$ values left and therefore $(n-2)!$* ways to choose them.

12.4: The total 2-cycles is always equal to half the number of total 1-n permutations, so that makes the average count 0.5 or $\frac{1}{2}$.

12.5: Generally, the total k-cycles is always equal to $\frac{n!}{k}$, where $n!$ is the total permutation count.

12.6: On average, each cycle appears $\frac{1}{\text{length}}$ times in each permutation, so $\frac{(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\cdots+\frac{1}{n})}{n}$ is the average of all lengths over any permutation of n.

12.7: If a permutation had two cycles of $> \frac{n}{2}$ length, then the total number of terms in the cycle decomposition would add up to $> n$, which is impossible because a permutation of n always has n terms. Likewise, any k cycles of $\frac{n}{k+1}$ in length must already add up to the full cycle decomposition of n terms, so k cycles of length $> \frac{n}{k+1}$ is impossible by the same reasoning as above.

12.8: Each prisoner has a $\frac{1}{2}$ chance of getting their paper, so all 100 prisoners have a total $\frac{1}{2^{100}}$ chance of going free. Not any time soon!

12.9: To have any higher chance of sucess, the prisoners have to use an extra piece of information. But the prisoners cannot talk, and only one is in the room at any time. In fact, there is only one main piece of information we know: Which paper is in which box, for up to 50 boxes. So a possible strategy for the prisoners to follow relies on the laws of cycle decompositions: Start by picking the box with your number on it. Then, look at the number inside and [win or] go to the box with that number on it. Keep doing this until you find your number or you lose. When you do this you create a cycle with your number in it because your first pick was the box... (cont.)

12.9 (cont.): with your number. In the worst-case scenario, there are 2 cycles of length 50 or one cycle of length 100, both of which give you a 50% chance of winning. Every other random case gives you a >50% chance of winning, because of the law from 12.7. Overall, this increases the prisoner's odds from $\frac{1}{2^{100}}$ to a much greater value, still under 50% but nonetheless greater odds. (Not sure of the exact constant, but it's somewhere between $\frac{1}{2^{100}}$ and 0.5!)

12.10: Start with just a 2-seater. Either person 1 picks your seat, or his own seat. This way you have a half ($\frac{1}{2}$) probability of getting your assigned seat. What about a 3-seater? ...(cont.)

12.10 (cont.): Either...

- The person 1 picks seat 1 - Win
- The person 1 picks seat 2 - 50/50 chance
- The person 1 picks seat 3 - Lose

These also yield a $\frac{1}{2}$ chance of getting your seat. Overall, the same method applies to 100 seats - I'm just not going to calculate it in time. Once person 1 chooses their seat, it sets off a cycle that either collapses (whoever was blocked most recently sits in seat 1) or continues until you are stuck. Both are equally likely because in reality one person must make the choice between seat 1 and seat 100, a 50/50 chance of sucess or doom. Or, just remember your boarding pass before you leave!

12.11: To have 1 and two in the same cycle, they must have been swapped. So, because each swap creates two equally likely cases of permutations (where $n \geq 2$), the probability must be $\frac{1}{2}$ or 50%.

12.12: Start by picking a random number for slot 1. Then, pick a random number for the slot of the random number in slot 1. Keep doing this until either you see 2 and 3 and 1 or until you see 1 and then go back to the start. Because the probability of a 3-cycle for any n is $\frac{1}{3}$, the probability that you see 2 and 3 before 1 (and ensure they are in the same cycle) is also simply $\frac{1}{3}$ or 33.333%.