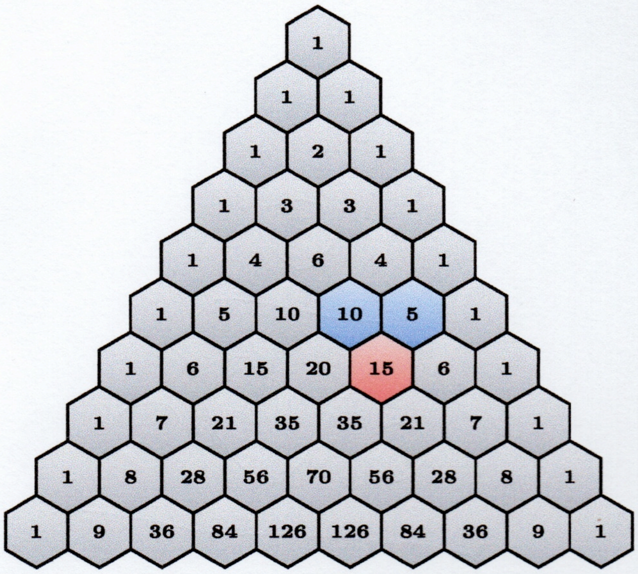
WORKSHEET 8

Pascal's Triangle

Pascal's triangle is a triangular array of numbers with 1's down the sides, and each number in the middle is the sum of the two numbers above it. The first few rows of Pascal's triangle are shown in Figure 1. Observe that the number in the red cell is equal to the sum of the numbers in the two blue cells.



1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

Figure 1. Pascal's Triangle.

PROBLEM 8.1. The numbers in the leftmost diagonal are all clearly equal to 1, and the numbers in the next diagonal are just the positive integers: $1, 2, 3, 4, 5, \ldots$ Can you find a formula for the numbers in the next diagonal? (That is, the one beginning $1, 3, 6, 10, 15, \ldots$) What about the next diagonal, beginning $1, 4, 10, 20, 35, \ldots$?

PROBLEM 8.2. What is the sum of the numbers in the n^{th} row of Pascal's triangle? (We consider the top row, consisting of just a 1, to be row 0, and then the row with two 1's is row 1, and so forth.)

If n is a positive integer, we define $n!$, pronounced "*n factorial*," to be the product of all the positive integers up to n, so $1! = 1$, $2! = 2$,

$3! = 3 \times 2 \times 1 = 6$, $4! = 4 \times 3 \times 2 \times 1 = 24$, and so on. We also define $0! = 1$.

PROBLEM 8.3. Show that the k^{th} entry in the n^{th} row of Pascal's triangle is $\frac{n!}{k!(n-k)!}$. (Both rows and diagonals are indexed starting from 0, so the red cell is in row 6 and diagonal 4, for instance.)

DEFINITION. The numbers $\frac{n!}{k!(n-k)!}$ are very important. We call them *binomial coefficients* and write them as $\binom{n}{k}$.

PROBLEM 8.4. Find the sum of the first m numbers in the r^{th} diagonal of Pascal's triangle, as a function of m and r.

Hockey Stick Identity

PROBLEM 8.5. What is the *alternating sum* of the numbers in the n^{th} row of Pascal's triangle? The alternating sum is the same as the sum, but with every other term subtracted. For instance, the alternating sum in row 4 is $1 - 4 + 6 - 4 + 1$. Explain your answer.

Always 0 - show that if row n is 0, then row n+1 is recursively 0.

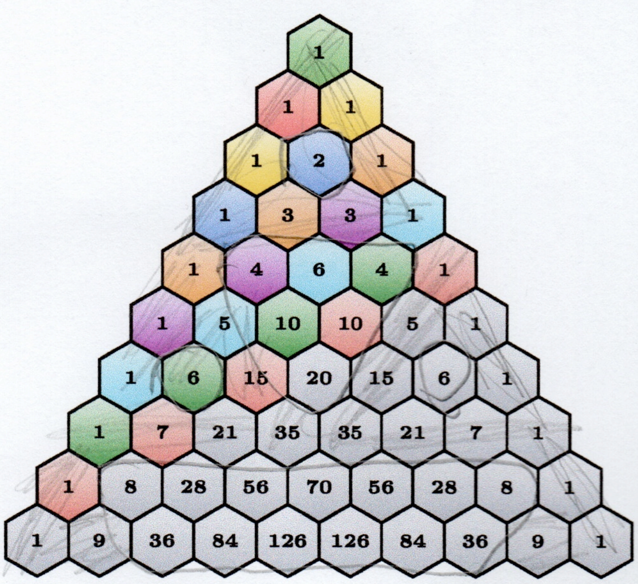


Figure 2. Shallow diagonals in Pascal's triangle.

PROBLEM 8.6. Sum the numbers in each monochromatic shallow diagonal in Figure 2. What familiar sequence of numbers do you get? Express this as a formula, using either factorials or the binomial coefficient notation. Why is your formula correct? Why do these numbers appear in Pascal's triangle?

If orange 1, 3, 1 is Fibonacci and purple 1, 4, 3 is Fibonacci, then orange plus purple equals cyan, which is Fibonacci. See it?

The Binomial Theorem

PROBLEM 8.7. Expand $(x + y)^4$ and $(x + y)^5$. Do you recognize these coefficients? Why is this happening? Does it work in general?

PROBLEM 8.8. What are 11^3 and 11^4? What's going on? Why does the pattern break down when you compute 11^5? What happens

$1 = 11^0$
$11 = 11^1$
$121 = 11^2$
$1331 = 11^3$
$14641 = 11^4$

No more 11! But...

$1\underline{0}5\,10\,10\,\underline{0}5\underline{0}1 = 101^5$

and

$1(10^5)+5(10^4)+10(10^3)+10(10^2)+5(10^1)+1(10^0)=11^5$!

when you compute 101^5 and 101^6? Explain why this happens. Where does this pattern break down, and how can you get a longer-lasting version of this phenomenon?

PROBLEM 8.9. Start at the top cell of Pascal's triangle, and at every second, move either one cell southwest or one cell southeast. How many paths are there that end up at the k^{th} diagonal in the n^{th} row?

waterpath Method for counting

PROBLEM 8.10. Draw a large Pascal's triangle, and shade in all the cells containing odd numbers. What pattern do you get? Can you explain what's going on?

Serpenski's

PROBLEM 8.11. The number 1 clearly appears infinitely many times in Pascal's triangle, but every other number only appears finitely many times. The number 10 appears 4 times. Can you find a number other than 1 that appears more than 4 times? More than 6 times? What is the largest number of occurrences you can find?

Hint:

Largest # happens 8 times

If x happens at least twice,

$$x = \binom{x}{1} = \binom{x}{x-1}$$

$120 = \binom{120}{1} = \binom{16}{2}$ → Look, more than one way $\binom{x}{1} = \binom{x}{x-1}$!

Reason why counting binom. coefficients works

Find each number. (4 times, 6 times) < 150

(8 times) = good luck!

*short for "Triangle," Not delta.

Pascal's Awesome Triangle

Alexander Friesen 11/11-16/25

8.1: Row by row, here we go:

Row 0: It's just ones. No exceptions.

Row 1: 1, 2, 3, 4, 5, 6, 7... "counting numbers."

Row 2: 1, 3, 6, 10, 15, 21, 28... triangular numbers!

Row 3: 1, 4, 10, 20, 35, 56, 84... each term is the sum of the previous term and the term's corresponding triangular number. These are called "tetrahedral."

Row 4: 1, 5, 15, 35, 70, 126... Same as Row 3 but adding tetrahedral numbers each time instead. These are "pentatope." At this point, four dimensions are used to express "pentatopes" geometrically.

This pattern continues infinitely.

8.2: The sums are 1, 2, 4, 8, 16... $= 2^0, 2^1, 2^2, 2^3, 2^4... = \boxed{2^n}$ for each row n!

8.3: A crucial part of Pascal's Δ^* is that every ${}^{n}{}_{K} + {}^{n}{}_{K+1} = {}^{n+1}{}_{K+1}$. When ${}^{n}{}_{K}$ is the K'th term of the n'th row. (cont.)

8.3(cont.): Say that $n_k = \frac{n!}{(n-k)!k!}$ and $n_{k+1} = \frac{n!}{(n-k-1)!(k+1)!}$

Also, we want to prove that the sum of the two is $(n+1)_{k+1} = \frac{(n+1)!}{(n-k)!(k+1)!}$.

Algebra time: $\frac{n!}{(n-k)!k!} + \frac{n!}{(n-k-1)!(k+1)!} = \frac{(n+1)!}{(n-k)!(k+1)!}$

$$\frac{n!\cdot(k+1)}{(n-k)!(k+1)!} + \frac{n!\cdot(n-k)}{(n-k)!(k+1)!} = \dots$$

$$\frac{n!\cdot(k+1) + n!\cdot(n-k)}{(n-k)!(k+1)!} = \frac{(n+1)!}{(n-k)!(k+1)!}$$

$$n!\cdot(k+1) + n!\cdot(n-k) = (n+1!)$$

$$n!\cdot k + n! + n!\cdot n - n!\cdot k = (n+1)!$$

$$n! + n!\cdot n = (n+1)!$$

$$n!(1+n) = (n+1)!$$

$$n!\cdot(n+1) = (n+1)!$$

$$(n+1)! = (n+1)!$$

Checks out!

Note: Every Pascal's △ number can be expressed as $\binom{n}{k}$! Cool, right?

8.4: If these m numbers in the rth diagonal can be expressed as $1, n_1, n_2, \dots, n_{m-1}, n_m$, the sum of all of these numbers is n_m in the diagonal $r+1$. This is called the Hockey Stick Identity because of how it looks in Pascal's △: (cont.)

*in an alternating sum

8.4(cont.):

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 $\boxed{35}$ 21 7 1

See the hockey stick (or sock?) In the row 4, 4 numbers 1+4+10+20 sum to the 4th number in row 5, which happens to be 35.

8.5: For every even-numbered row, this is simple - everything cancels. For each odd-numbered row, it can be expressed as the sum of every number above it, which can be rearranged to form two symmetrical copies of the previous row, which both equal 0*. This means that each even-numbered row translates its alternating sum of 0 to each odd-numbered row, so every row's alternating sum equals 0.

8.6: These sums are 1, 1, 2, 3, 5, 8, 13, 21, 34... ...that looks like the Fibonacci numbers! In fact, each monochromatic shallow term in figure 2 is the sum of the respective terms in the two previous diagonals. And because each Fibonacci number is the sum of the previous two, each monochromatic diagonal will always be a Fibonacci number because it is the sum of the previous two Fibonacci numbers.

8.7: $(x+y)^0=1$

$(x+y)^1=1x+1y$

$(x+y)^2=1x^2+2xy+1y^2$

$(x+y)^3=1x^3+3x^2y+3y^2x+1y^3$

$(x+y)^4=1x^4+4x^3y+6x^2y^2+4xy^3+1y^4$

$(x+y)^5=1x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+1y^5$

Now, look at only the coefficients:

1
11
121
1331
14641
15101051

It's Pascals triangle! (cont.)

8.7(cont.): This is called the Binomial Theorem, and it is very useful to find binomial coefficients quickly. A nice proof of this comes from 8.9, in which the number of "ways to get to a term" can be represented as the number of different binomial expansion terms that have the same variables - thus grouping them together into coefficients resemblant of Pascal's triangle.

8.8: $11^0 = 1$
$11^1 = 1\ 1$
$11^2 = 1\ 2\ 1$
$11^3 = 1\ 3\ 3\ 1$
$11^4 = 1\ 4\ 6\ 4\ 1$

Hey, it's Pascal's triangle again... wait... $11^5 = 161051$... but if we take row 5's dauble-digit terms and overlap them, we get: $15_{\substack{+\\1}}0_{\substack{+\\1}}051 = 161051 = 11^5$! (cont.)

8.8(cont.): This overlap rule works for 11^6 and beyond, keeping the 11^n pattern alive. As for $(101^5, 101^6 \ldots)$: $101^5 = \underline{1}0\underline{5\,10\,10}0\underline{5}0\underline{1}$, $101^6 = \underline{1}0\underline{6\,15\,20\,15}0\underline{6}0\underline{1}$. Pascal's △ is here too, it's just that a digit 0 is inserted infront of the first 1, before the last 1, and before the second-to-last term. However, even this pattern breaks at 101^7 and I'm not sure what algorithm works here (it's not the overlap one.)

8.9: A really cool property is that every $^{n}{}_{K}$ term is also the number of paths that contain no detours to get from the apex $0_0 = 1$ to n_K, moving down between terms left or right each time. This even holds in a grid:

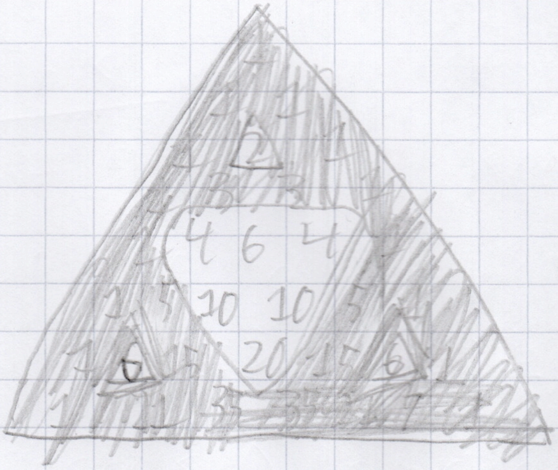
Start

1	1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10	11
1	3	6	10	15	21	28	36	45	55	66
1	4	10	20	35	56	84	120	165	220	286
1	5	15	35	70	126	210	330	495	715	1001
1	6	21	56	126	252	462	792	1287	2002	3003
1	7	28	84	210	462	924	1716	3003	5005	8008

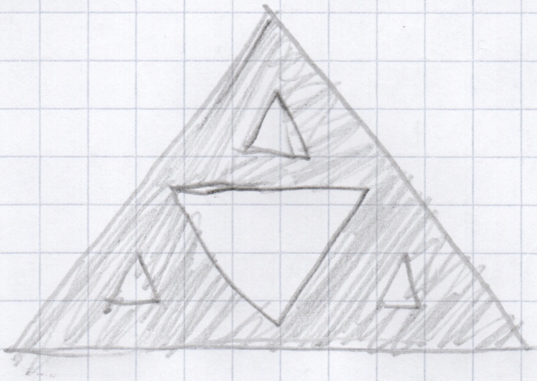
Finish

→ There are 8008 paths if the grid numbers are represented by Pascal's △.

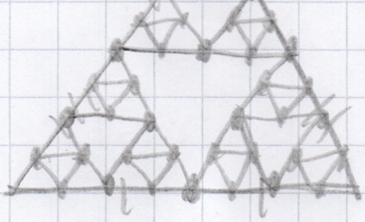
8.10:



Don't see it yet? Let's remove the numbers:



This creates the fractal Serpenski's Δ, a beautiful pattern made by infinitely connecting the midpoints of an equilateral triangle:



(of course, the exact pattern only forms when the 1s are removed from Pascal's Δ.)

8.11: What I know:

- Terms $n_k = \binom{n}{k} = nC_k$
- Pascal's Δ is symmetric
- (obviously) Each term is the sum of the above two

Where I'm stuck:

- Binomial $\binom{n}{k}$ how?
- Soooooo I learned that apparently brute force is basically the only way other than just conjectures and that some kids went as far as using PYTHON for 8.11?!

What I ended up looking up:

- 10 is the smallest number that appears four times
- 120 for 6 times
- and 3003 is the *only* known number that appears eight times - so far.
- All this was found via brute force (computerized) algorithms that I really don't have the time, energy, or motivation to compute.
- "Just Count It." - Internet user.
- Singmaster's Conjecture of a finite upper bound for this problem.