WORKSHEET 13

Invariants and Monovariants

PROBLEM 13.1. Suppose we start with an $m \times n$ bar of chocolate that we wish to divide into individual 1×1 squares. On every turn, we cut one of the pieces into two by breaking it along one of the horizontal or vertical grid lines. What is the smallest number of breaks we have to do to break the entire bar into 1×1 squares?

PROBLEM 13.2. The numbers from 1 to 1000 are written on the board. At every turn, we take two of the numbers, say a and b, erase them, and replace them with the number $a + b - 1$. After 999 steps, we are left with a single number. What are the possibilities for this number?

PROBLEM 13.3. The numbers from 1 to 1000 are written on the board. At every turn, we take two of the numbers, say a and b, erase them, and replace them with the number $|a - b|$. After 999 steps, we are left with a single number. Could this single number be 3?

PROBLEM 13.4. We have a 10×10 grid of lights. Initially, the light in the top left cell is on, and all the other lights are off. At each step, we pick a row or column of the grid and toggle all the lights in that row or column. Is it possible to perform some sequence of these steps such that all the lights are turned on at the same time?

PROBLEM 13.5. You have a quarter infinite (to the right and up) grid, with checkers initially placed at the three corner squares, as shown in Figure 1. At every step, if there is a checker at the cell (x, y), you can remove it and replace it with a checker at $(x, y+1)$ and a checker at $(x+1, y)$, provided that neither of these two cells already has a checker. Is it possible to perform some sequence of moves such that the three cells that initially contained checkers are empty?

PROBLEM 13.6. Start with a finite collection $a_1, a_2, \ldots, a_n$ of positive integers. On each turn, you choose two numbers a_j and a_k such that neither one is a multiple of the other, and then delete them both and replace them with $\gcd(a_j, a_k)$ and $\operatorname{lcm}(a_j, a_k)$.[1] If there is no pair

[1] $\gcd(a, b)$ is the greatest common divisor: the largest number that divides both a and b. $\operatorname{lcm}(a, b)$ is the least common multiple: the smallest number that is a multiple of both a and b.

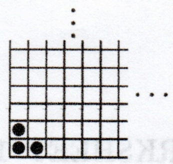

Figure 1. Three checkers placed at the bottom left of a quarter-infinite chessboard.

(a_j, a_k) such that neither is a multiple of the other, then the game ends. Is it possible to keep playing forever?

As you solved these problems, you probably noticed that they have something in common: some quantity that doesn't change when you perform an operation. This quantity is called an *invariant* for the process.

Sometimes, we find a quantity that doesn't stay the same at every step of the process, but it only changes in one direction: it can only increase, or only decrease. In that case, the quantity is called a *monovariant*.

PROBLEM 13.7. Suppose we have n red points and n blue points in the plane, with no three points collinear. Show that it is possible to draw n nonintersecting line segments, each connecting one red point and one blue point.

PROBLEM 13.8. We have a 10×10 grid, with each cell containing a strawberry. Initially, 9 of the strawberries are moldy. If at any time, a strawberries has two neighboring (either horizontal or vertical) moldy strawberries, then it becomes moldy as well. Is it possible for the mold to spread to all the strawberries?

We conclude with a walkthrough of a remarkable monovariant problem called *Conway's checkers* or *Conway's soldiers*. We have an infinite (in all directions) grid, with the center of each grid point having the form (m, n), where m and n are integers. We initially have a checker at each point (m, n) where $n < 0$, as shown in Figure 2.

On each turn, if we have two checkers on adjacent squares, and the next square is empty, then we can make a capture, as shown in Figure 3. Capturing can be done in the horizontal direction or the vertical direction. The challenge is to perform a sequence of captures to get a checker as high up the board as possible.

PROBLEM 13.9. Find a sequence of captures to get a checker to the point $(0, 3)$. (That is, four rows above the top row of checkers in the initial position.)

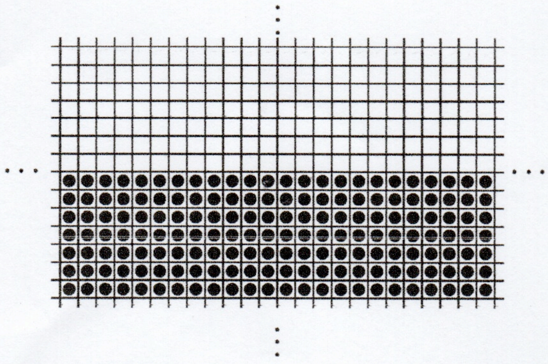

Figure 2. The initial position of Conway's checkers.

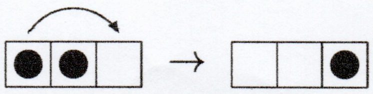

Figure 3. Capturing in Conway's checkers: if we have checkers on the two black squares on the left, and the next square is empty, then the leftmost checker can capture the one on the middle square, yielding the position on the right.

We will now show that it is not possible to reach the point $(0,4)$. Let $f(x,y)$ be a function, to be determined later. We wish to choose f such that if we make a capture from (x,y) to $(x,y+2)$ or $(x+2,y)$ that gets closer to $(0,4)$, then $f(x,y+2) = f(x,y) + f(x,y+1)$ or $f(x+2,y) = f(x,y) + f(x+1,y)$; in other words, getting closer to $(0,4)$ doesn't change the sum of the values of the checkers on the board. However, if we capture away from $(0,4)$, then the sum of the values should decrease.

PROBLEM 13.10. Is there a function of the form $f(x,y) = a^{x+y}$ with this property? If not, can you find a closely related function with this property?

PROBLEM 13.11. Now that you have found a suitable function f, find the sum of the values of all the checkers in the initial position, and the value of a hypothetical checker at $(0,4)$.

PROBLEM 13.12. Conclude that it is not possible to reach $(0,4)$ in finitely many captures.

Show that all checkers below line $<(0,4)$. Impossible to reach!

PROBLEM 13.13. Suppose we consider Conway's checkers in 3 dimensions, with checkers initially at all integer points (x,y,z) with $z<0$. What is the largest value of z for which it is possible to get a checker to $(0,0,z)$?

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13.1: You would have to break exactly through each indent once, no more, no less. This requires $n-1$ length-wise and $m-1$ width-wise breaks, for a chocolate bar n by m size.

13.2: Take, for example, 1-2-3-4-5. No matter the sequence of action, the result is always 11, which is also equal to $1+2+3+4+5-4$, the 4 being the sum of all -1's and the $1+\dots+5$ being the sum of all $a+b$'s. This pattern holds with 1-2...-1,000, where the <u>Only</u> possible result is $1+2\dots$ $\dots+1{,}000-999$, or 1,000 plus the sum of all integers from 1-998.

13.3: Here, when a and b are replaced with $|a-b|$, the new sum is $S-a-b+|a-b|$. This splits into 2 cases: $S-a-b+(a-b)$ and $S-a-b+(b-a)$. Both simplify... (cont.)

13.3(cont.):... into S-2b and S-2a. These are both even, and because S is even (500,500), 3 is impossible to end with because it is odd.

13.4: In any case of moves, the top-left light will always be the opposite of either the row or column of lights that it is in, so complete Uniformity is impossible.

13.5: Say that you are only allowed 1 checker, and every move splits the checker in half. The board:

1/64						
1/32	1/64					
1/16	1/32	1/64				
1/8	1/16	1/32	1/64			
1/4	1/8	1/16	1/32	1/64		
$\boxed{1/2}$	1/4	1/8	1/16	1/32	1/64	
$\boxed{1}$	$\boxed{1/2}$	1/4	1/8	1/16	1/32	1/64

The sum of the first 3 corner squares is 2, and the infinite sum of every other square turns out to also be 2. To win, this means that you have to fill ... (cont.)

13.5(cont.): every single square outside of the first 3 to match the sum, resulting in success taking infinitely long to achieve - basically impossible.

13.6: For any integers (a, b) where a is not a multiple/factor of b, $LCM(a,b) \cdot GCF(a,b) = a \cdot b$ and $LCM(a,b) \ldots$ $\ldots + GCF(a,b) > a+b$. In other words, when a move is made, the total sum of $a_1, \ldots, a_k$ increases while the total product does not change. Therefore, this sum would increase to ∞ as well as the highest number. Of course, no number can be made ∞ with a finite number of steps, so this is impossible to play forever.

13.7: It is always possible to divide this plane into a number of regions where each region contains 1 red and 1 blue point. Simply connect these points.

13.8: This is the best possible configurations for strawberries to be moldy:

M = moldy orig.

= turns until moldy

□ = safe from mold

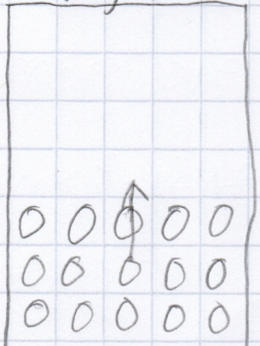
M	1	2	3	4	5	6	7	8
1	M	1	2	3	4	5	6	7
2	1	M	1	2	3	4	5	6
3	2	1	M	1	2	3	4	5
4	3	2	1	M	1	2	3	4
5	4	3	2	1	M	1	2	3
6	5	4	3	2	1	M	1	2
7	6	5	4	3	2	1	M	1
8	7	6	5	4	3	2	1	M

In any other configurations, the strawberries will have less diagonal connection, preventing their spread. In the best case, you need a 10th moldy strawberry at the ✳ to cover the whole grid in mold.

13.9: The minimal starting position uses 21 checkers:

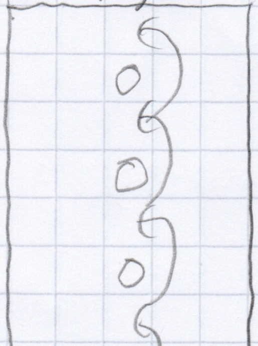
Eventually, moving every checker to the middle creates fig. 2: ...

fig 1



Starting.

fig2



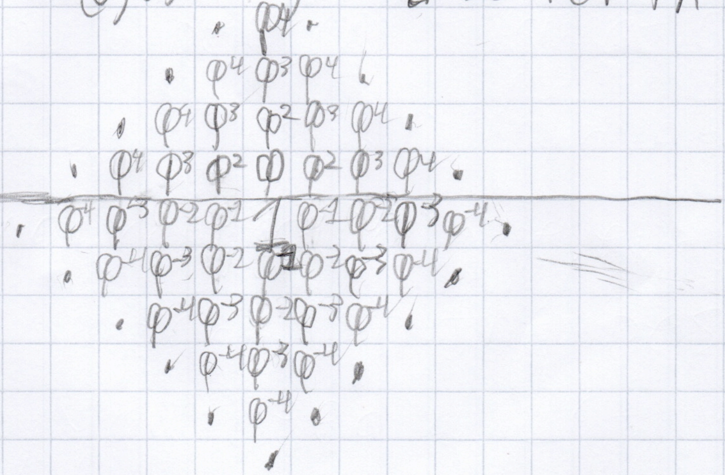
Finish.

*Only in $\frac{1+\sqrt{5}}{2}$ form. $\frac{1-\sqrt{5}}{2}$ is different.

13.9: (cont.): ... which allows the bottom 21st checker to move to (0,3).

13.10: This function would satisfy $a^{x+y}+a^{x+y+1}=a^{x+y+2}$. Dividing both sides by a^{x+y} gives $1+a=a^2$. This famous quadratic equation has the solution $a=\frac{1+\sqrt{5}}{2}$, or the golden ratio (φ).* So, $f(x,y)=\varphi^{x+y}$.

13.11: φ^{x+y} at $(0,0)=1$, and so forth‡:



The (0,4) checker, or (0,5) in the graph above, has a value of φ^5. Each square below the line has a negative power, and above the line a positive power.

‡This graph represents the "feasability" of getting a checker from 1 to φ^x.

13.12: The sum of the entire space below the line is $1+3\cdot\varphi^{-1}+5\cdot\varphi^{-2}\ldots$ $\ldots+7\cdot\varphi^{-3}+9\cdot\varphi^{-4}\ldots$ and untill $\infty\cdot\varphi^{-\infty}$. This is both an arithmetic and a geometric series, so the sum formula is $S=\frac{a_1}{1-r}+\frac{dr}{(1-r)^2}$, where a_1 = 1st term, d = common difference, and r = common ratio. Substituting: $S=\frac{1}{1-\varphi^{-1}}+\frac{2\varphi^{-1}}{(1-\varphi^{-1})^2}$. $S=\frac{(1-\varphi^{-1})+2\varphi^{-1}}{(1-\varphi^{-1})^2}$. Because $1+\varphi=\varphi^2$, $1-\varphi^{-1}=\varphi^{-2}$. So $S=\frac{\varphi^{-2}+2\varphi^{-1}}{\varphi^{-4}}=\ldots$ $\ldots=\varphi^2+\frac{2\varphi^{-1}}{\varphi^{-4}}=\varphi^2+2\varphi^3=\varphi^2(1+2\varphi)$. $\varphi^2=1+\varphi$, so $\varphi^2((1+\varphi)+\varphi)=\varphi^2(\varphi^2+\varphi)$, $=\varphi^4+\varphi^3$, which equals φ^5.

That means that to get to space (0,5) (or (0,4)), you need to have an infinite number of checkers below the line – impossible to win with a finite number of moves!

13.13: In 3 dimensions, the sum of the values of all checkers behind the (x-y) plane is larger:

$$1+5\varphi^{-1}+13\varphi^{-2}+25\varphi^{-3}+41\varphi^{-4}+61\varphi^{-5}\ldots$$

Each coefficient is represented by the nth centered square number: $a_n=(2n^2-2n)+1$.

$$S=\sum_{n=0}^{\infty}(2n^2-2n+1)r^n=2\left(\frac{r(1+r)}{(1-r)^3}\right)+2\left(\frac{r}{(1-r)^2}\right)+\frac{1}{1-r}$$

where r = common ratio $=\varphi^{-1}$ and n = term index for a_n.

$$S=\frac{2r+2r^2}{(1-r)^3}+\frac{2r}{(1-r)^2}+\frac{1}{1-r}=\frac{2r+2r^2+2r-2r^2+1-2r+r^2}{(1-r)^3}$$

$$=\frac{r^2+2r+1}{(1-r)^3}=\frac{(1+r)^2}{(1-r)^3}=\frac{(1+\varphi^{-1})^2}{(1-\varphi^{-1})^3}=\frac{\varphi^2}{\varphi^{-6}}\ldots$$

... which simplifies to φ^8.

So, in 3 dimensions you can get up to 7 spaces out, but on (0,0,8) it becomes impossible.