WORKSHEET 3

Impartial Combinatorial Games

Consider the game of MATCHES.[1] In this game, there is a pile of matches, with n matches in the pile initially. Two players take turns removing some matches from the pile, and the player to move can remove 1, 2, or 3 matches on a turn. The player who is not able to make a move (because the pile is gone) loses the game.

PROBLEM 3.1. For $1 \leq n \leq 20$, determine whether the first or second player has a winning strategy at MATCHES, assuming that both players play optimally. What about for all n? Can you explain your observation?

PROBLEM 3.2. What happens if the person who takes the last match loses rather than wins?

REMARK. In combinatorial game theory, the rule that the player who is not able to make a move loses is called the *normal-play convention*, whereas the rule that the player who is not able to make a move wins is called the *misère-play convention*. Most of the time, the misère-play games are much more difficult to study than normal-play games, but in the case of MATCHES, both have simple analyses.

PROBLEM 3.3. Consider a generalization of MATCHES, where a player can remove up to k matches on each turn. Now, for which values of n does the first player have a winning strategy in normal play? What about in misère play? Explain why your answers are correct.

We call these types of games *subtraction games*. If S is a set of positive integers, we write SUBTRACTION$_S$ for the subtraction game where the next player can remove any number of matches from S. For instance, the original game is SUBTRACTION$_{\{1,2,3\}}$.

PROBLEM 3.4. Analyze the games of SUBTRACTION$_{\{1,2,4\}}$ and SUBTRACTION$_{\{1,3,4\}}$.

Suppose we fix a subtraction set S and consider the game SUBTRACTION$_S$. For a pile size of n, we say that n is an $\mathcal{N}$ *position* if

[1]It is a common convention to use SMALL CAPS for the names of games in combinatorial game theory.

the next player to move has a winning strategy with optimal play, and a $\mathcal{P}$ *position* if not. (The $\mathcal{P}$ stands for "previous," since we imagine the game has been going on for a while, so the player other than the next player must have been the previous player.) We can consider a sequence $a_0, a_1, a_2, \ldots$, where $a_n = \mathcal{N}$ or $\mathcal{P}$, depending on whether n is an $\mathcal{N}$ position or a $\mathcal{P}$ position, respectively.

DEFINITION. Given a sequence $a_0, a_1, a_2, \ldots$, we say that it is *periodic* if it repeats indefinitely, i.e. there exists a positive integer p such that $a_{n+p} = a_n$ for all $n \geq 0$. We say that it is *preperiodic* if it eventually repeats indefinitely, i.e. there exists a positive integer p and a nonnegative integer n_0 such that $a_{n+p} = a_n$ for all $n \geq n_0$.

PROBLEM 3.5. Show that for any finite set S of positive integers, the sequence a_n corresponding to SUBTRACTION$_S$ is preperiodic.

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PROBLEM 3.6. Find a finite subtraction set S for which the sequence a_n is not periodic.

For a preperiodic sequence a_n, the *period* is the smallest positive integer p such that there exists a nonnegative integer n_0 such that $a_{n+p} = a_n$ for all $n \geq n_0$.

PROBLEM 3.7. Suppose $S = \{a, b\}$. In terms of a and b, determine the period of SUBTRACTION$_S$. (Try a bunch of examples before trying to guess the answer!)

REMARK. The answer to the previous problem is unknown in general if $S = \{a, b, c\}$ contains three numbers.

PROBLEM 3.8. Consider the following game between two players:[2] we start with a pile of n stones. The first player can remove as many stones as desired, but at least one and not the entire pile. On subsequent turns, the next player can only take at most as many stones as the previous player took. The player who is unable to make a move (usually because there are no stones left, but also at the beginning when $n = 1$) loses. For which values of n does the first player have a winning strategy?

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PROBLEM 3.9. Answer the same question as in the previous problem, but now each player can take up to twice as many stones as the previous player took.

PROBLEM 3.10. Consider the following game, known as WYTHOFF'S GAME: there are two piles of stones. The player to move can either remove as many stones as desired from one pile, or can remove an equal number of stones from both piles. What are the $\mathcal{N}$ and $\mathcal{P}$ positions of

when are
piles ≠,
when are
piles = ?
Two cases.

[2] I'm not going to tell you the name of this game, because the name gives away the answer.

this game? (You might not be able to give a complete answer to this problem, but figure out as much as you can!)

The game of NIM is played as follows: there are several piles of stones, and the player to move can take as many stones as desired, but only from one pile per turn. We represent a position with k piles, of size $b_1, b_2, \ldots, b_k$ as $(b_1, b_2, \ldots, b_k)$. While there is a beautiful general winning strategy for NIM, coming up with it on your own is not an easy task. So instead, we'll look at a few special cases.

PROBLEM 3.11. If there is only one pile of size b_1, when is this game an $\mathcal{N}$ position, and when is it a $\mathcal{P}$ position?

PROBLEM 3.12. Consider the NIM game (b_1, b_2) with two piles. When is this an $\mathcal{N}$ position? When is it a $\mathcal{P}$ position?

simple

PROBLEM 3.13. Find at least 10 triples (b_1, b_2, b_3) of positive integers with $b_1 < b_2 < b_3$ such that (b_1, b_2, b_3) is a $\mathcal{P}$ position. Can you find triples of many different types? What are some general patterns you can come up with?

↳ use the "x2 also works" strat to

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3.1: The chart below shows the pattern:

n	1	2	3	4	5	6	7	8	9	10
who wins	1st	1st	1st	2nd	1st	1st	1st	2nd	1st	1st

...	11	12	13	14	15	16	17	18	19	20
	1st	2nd	1st	1st	1st	2nd	1st	1st	1st	2nd

Every multiple of 4 (including 0) is a "death trap" for whoever lands on it because the opponent can lead them to 0, making these 2nd-player wins. Every other position is a 1st-player win.

3.2: Same thing, but misère:

n	1	2	3	4	5	6	7	8	9	10
who wins	2nd	1st	1st	1st	2nd	1st	1st	1st	2nd	1st

...	11	12	13	14	15	16	17	18	19	20
	1st	1st	2nd	1st	1st	1st	2nd	1st	1st	1st

In misère play, every multiple of 4, +1, is a "death space," and everything else is a win for the 1st player.

3.3: Generalizations of MATCHES have the same pattern as 3.1 and 3.2, but replace 4 with k. In other words, in normal play every n which is a multiple of $k+1$ is a "death space" (explained on the first page) and in misère every n which is a multiple of $k+2$ $((k+1)+1)$ is a "death space." If $k=1$, where players can only pick 1 match, every other n is a "death space." If $k=2$, players pick from $\{0, 1\}$, and every third n is a "death space." $k=3$ was already shown in 3.1 and 3.2, and for each k, a natural number from 1 to ∞, the pattern holds.

3.4: The pattern here is: (1, 2, 4)

n	1	2	3	4	5	6	7	8	9	10
who wins	1st	1st	2nd	1st	1st	2nd	1st	1st	2nd	1st

Every multiple of 3 is a death space.

For (1, 3, 4):

n	1	2	3	4	5	6	7	8	9	10
who wins	1st	2nd	1st	1st	1st	1st	2nd	1st	2nd	1st

The pattern turns out to be 1st 2nd 1st 1st 1st 1st 1st, 1st 2nd ...

Note: N's are 1st-player starting wins, and P's are 2nd-player starting wins.

3.5: If you followed along with the method used to make the charts in problems 3.1 - 3.4, you may have noticed that it is possible to derive the entire sequence from a set of length x within it, where x is the largest subtraction number (because you can subtract from a_{x+1} all possible subtractions and determine if the term a_{x+1} is N or P, and x can keep increasing.) Because this algorithm can be used to determine the entire sequence, a pattern following the subtraction values in the set S will always show up - however it may take a few terms to get started like in $\text{SUBTRACTION}_{(1,3,4)}$. Therefore, every MATCHES sequence must be at least preperiodic, if not periodic also.

* Divisible by P, is a+b. The proof is insane so I didn't write it.

3.6: There are plenty of examples, one being the set $S_{\{1,3,4\}}$, which goes in a PNPNNNN pattern. However, the $n=1$ case is a 1st player win, and the sequence doesn't start at P like the recurring pattern does. This set (1,3,4) is an example of a preperiodic but not periodic finite subtraction set.

3.7: Here are periods for various (a,b)'s:

(1,2): 3 = p(NNP)	(1,3): 2 = p (NPNPNP...)
(2,3): 5 = p (PNNNP)	(2,4): 6 = p (PPNNNN)

For most of these examples, the set of N's has length b and the set of P's has length a because every N can be achieved by sending your opponent to a P using a subtraction of either b or a (because $a<b$ in this case.) In a few special cases such as (1,3) this pattern simplifies itself into a lesser p. Most of these examples have $p=a+b$ because of the previous reasoning, but a few have $p \neq a+b$.*

*nostalgia feeling from 3.9 to week 1

3.8: A table is a handy mathematical furniture:

n (size)	1	2	3	4	5	6	7	8	9	10
1st or 2nd wins	2nd (1st can't move)	2nd	1st	2nd	1st	2nd	1st	2nd	1st	2nd

On every odd number but 1, Player 1 wins, and elswhere Player 2 can easily claim victory. P1's dominant strategy on odd $\neq 1$ numbers is to take 1 stone every time, leaving Player 2 in a pickle.

3.9: Table again:

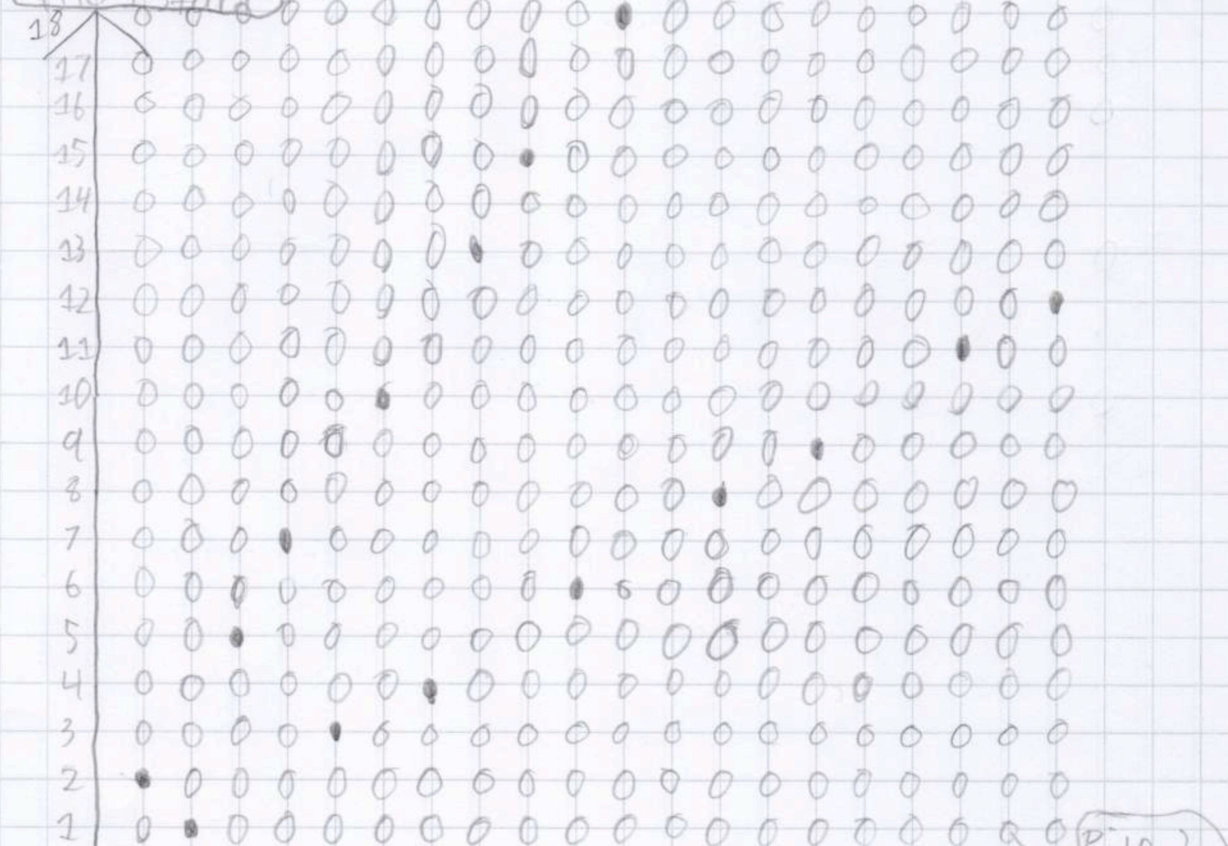
n (size)	1	2	3	4	5	6	7	8	9	10
1st or 2nd wins	2nd (1st can't move)	2nd	2nd	1st	2nd	1st	1st	2nd	1st	1st

It turns out that the 2nd player only wins on Fibonacci numbers (13 is the next 2nd-player win.) This makes sense because each Fibonacci number is the sum of the previous two, and the gaps between each 2nd-player win double each time from $3 = n$. So, the answer is F_n*?

*where $x > 0$, of course.

3.10: A systematic approach is best. When both piles are equal, the first player always wins by setting them to $(0,0)$, leaving the second player hopeless.

Any (x,x) position* is N by this strategy. However, these aren't the only N positions, and finding the P positions is even trickier. Let's generalize this with a graph of N and P positions:



N and P positions for starting points. $N = 0$, $P = \bullet$. Pile 1 starting tokens $= y$, and Pile 2 starting tokens $= x$. (cont.)

3.10 (cont.): Some observations:

- Any N=0 position can be reached by a P=• position using cardinal directions (N W E S).
- No P position can reach eachother with a queen's move in chess (This actually inspired another game called "Wythoff's" that uses a queen.)
- All P positions on the same NW-SE line grow consecutively in distance
- It turns out that the slope of the lines created by the P-points is the golden ratio and its reciprocal! (This was looked up on Google, as it includes higher mathematics.)

3.11: One pile is extremely boring, as the first player takes it all and gets an N position every single time. No P positions.

3.12: NIM here gets more complicated, so cases are a better idea.

Case 1: 0 in 1 pile. This is 3.11!

Case 2: 1 in 1 pile. Always go first and take all but one in the second pile.

(cont.)

*Note that if the 2nd pile is the same as the first it is P.

3.12 (cont.):

Case 3: 2 in 1 pile. Take all but two in the second pile as the first player to guarantee a win. If there is <2 in the second pile, go to Cases 1 and 2.

Case 4: 3 in 1 pile. Take all but three in the second pile as the first player to guarantee a win. If there is <3 in the second pile, go to Cases 1, 2, and 3.

I think a pattern is emerging... In fact, these cases can be used to determine all N's and P's just from the strategy for each!*

3.13: You may have noticed from 3.12 that:

- All P positions have equal-sized piles
- Everything else is a N position.

It turns out that (after some research) binary addition without carrying proves why this is true.

(cont.)

3.13(cont.): Let's try the binary "addition without carry" on an N position and a P position:

For $N(2,3)$:

$$\begin{array}{rr} 2 = & 10 \\ 3 = & 11 + \\ \hline 1 = & 01 \end{array}$$

For $P(3,3)$:

$$\begin{array}{rr} 3 = & 11 \\ 3 = & 11 + \\ \hline & 00 \end{array}$$

If there is an even number of 1's in a column, the result is 0, and elsewhere it is 1. A P position has the key factor of always summing to 0. That is why only (x,x) positions in two-pile NIM are P: because their binary digits line up perfectly to get a 0 sum.

We can use this observation to find 3-pile NIM P and N positions.

For example, $(1,2,3)$ is a classic P position:

$$\begin{array}{rr} 1 = & 1 \\ 2 = & 10 \\ 3 = & 11 + \\ \hline & 00 \end{array}$$

00 means that $(1,2,3)$ is P.

But instead of just hunting for possibilities, another method can speed the process:

If a binary number is doubled,

$(3 \cdot 2 = 11 \cdot 2 = 110 = 6,\ 7 \cdot 2 = 111 \cdot 2 = 1110 = 14$, etc.)

a 0 is put on the end of the number to create a new number (like x10 in base 10.)

(cont.)

* Some other unique P positions (looked up) are (6,10,12), (20,39,51)

13.3 (cont.): Using this observation, it is possible to determine all P positions to an infinite level in NIM because another column of 0's keeps the sum 0, as in the (1,2,3)→(2,4,6) piles:

$$\begin{array}{r} 2 = 10 \\ 4 = 100 \\ 6 = 110 + \\ \hline 000 \end{array} = \begin{array}{r} 1 = 1 \\ 2 = 10 \\ 3 = 11 + \\ \hline 00 \end{array}$$

$$= \quad =$$

$$\begin{array}{r} 4 = 100 \\ 8 = 1000 \\ 12 = 1100 + \\ \hline 0000 \end{array}$$

In fact, we can do this for any unique* P position!