WORKSHEET 16

Identification Spaces and Surfaces

If you take a square and glue the opposite edges together, you get a cylinder, as shown in Figure 1.

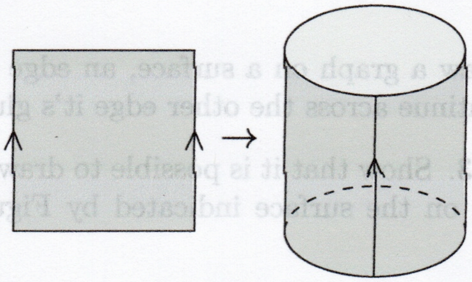


Figure 1. Gluing opposite edges of a square yields a cylinder.

Problem 16.1. What happens when you glue the opposite edges of the cylinder? Equivalently, what happens when you glue the edges as marked in Figure 2? Note that you can't do this with paper, but you can do it using a stretchy material.

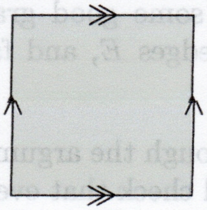


Figure 2. What happens when you glue the marked edges here?

Problem 16.2. What happens when you glue the marked edges in Figure 3?

A figure like the one in Figure 3 is called an *identification space.* An identification space consists of a polygon, together with instructions for how to glue the edges.

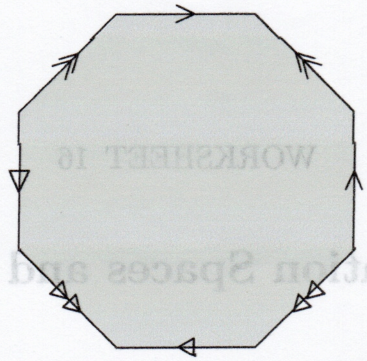


Figure 3. An octagon with edges ready to be glued.

When you draw a graph on a surface, an edge can cross over the boundary and continue across the other edge it's glued to.

PROBLEM 16.3. Show that it is possible to draw K_5 and $K_{3,3}$ with no crossing edges on the surface indicated by Figure 2, known as a *torus.*

PROBLEM 16.4. More generally, we have the complete graph K_n, which consists of n vertices and edges between every pair. Show that it is possible to draw K_6 and K_7 with no crossing edges on a torus.

The most natural graphs to draw on a torus are connected graphs such that each face can be continuously shrunk to a point, i.e. they don't go around any "hole" of the torus. We will call such graphs *good* graphs; this isn't a standard term, so its scope will be limited to this worksheet.

PROBLEM 16.5. Draw some good graphs on a torus and count the number of vertices V, edges E, and faces F. Do you notice any patterns?

PROBLEM 16.6. Go through the argument showing that for planar graphs, $V - E + F = 2$, and check that everything still works for good graphs on a torus.

PROBLEM 16.7. Show that we cannot draw K_8 on a torus with no crossing edges.

Since the number $V - E + F$ of a good graph on a surface is independent of the choice of good graph, it is an invariant of the surface alone. We call it the *Euler characteristic* of a surface.

PROBLEM 16.8. What is the Euler characteristic of the surface shown in Figure 3? Work with the easiest good graph possible.

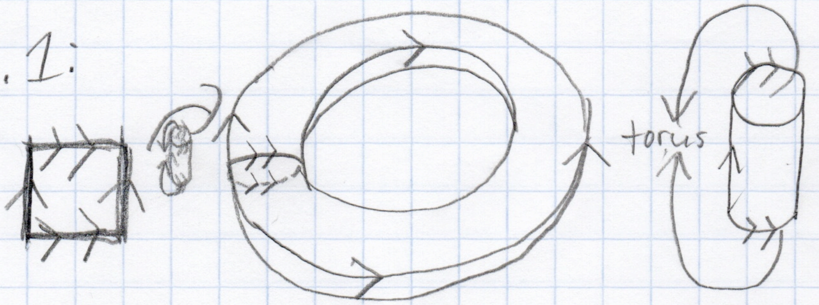
Problem 16.9. Is it possible to glue the edges of the octagon differently, still in pairs, to get a different Euler characteristic? What values can you get?

Problem 16.10. By now, you presumably know that the octagon in Figure 3 is a double-holed torus. Find an identification space for a triple-holed torus, and more generally for an n-holed torus. What is the Euler characteristic of an n-holed torus?

Identification Spaces and Surfaces

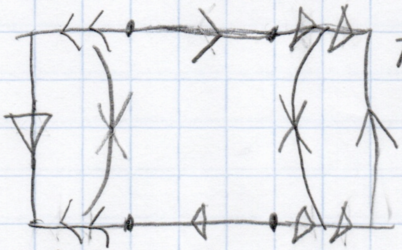
Alexander Friesen 2/17-22/26

16.1:



This is a donut-shaped piece of paper that has been moved in the third dimension, called a torus. To create it, first fold a rectangle into a cylinder, then wrap its two bases (marked by >>) on to eachother such that the warped cylinder only has one face - a torus.

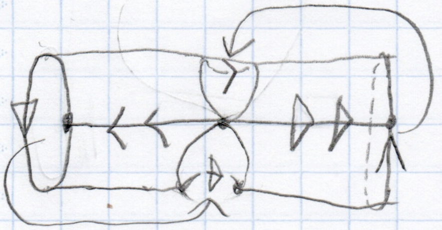
16.2: Start by reforming the octagon into:



A rectangle

Then:

Fold << and ▷▷.



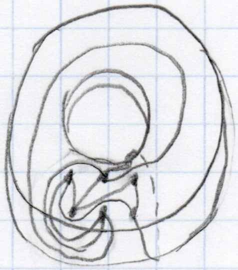
A cylinder with two holes

Finally: Fold each hole on to the other of the same type.

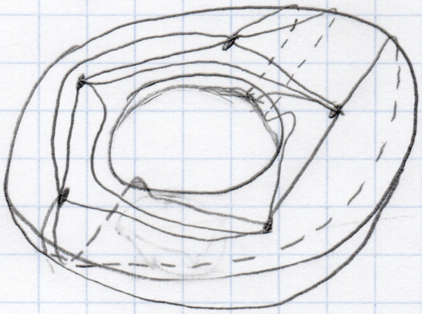
Two Holed Torus:

A gen. 2 torus

16.3: Draw as much of a $K5$ or $K_{3,3}$ as possible, and when the intersection pops up carry the line around the hole. Ex:



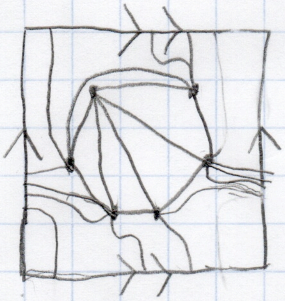
$K_{3,3}$



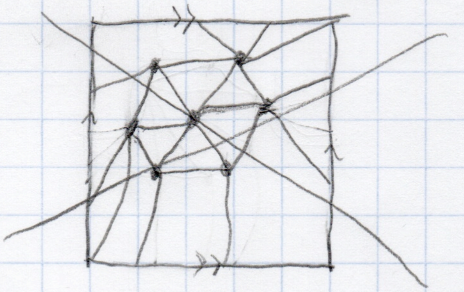
K_5

Note: All torii will be drawn as  from now on.

16.4:



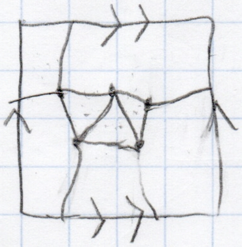
K_6 on a Torus



K_7 on a Torus

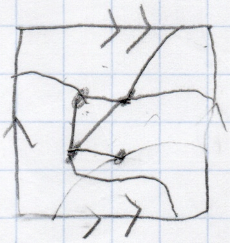
I got stuck. It's too complicated!

16.5:



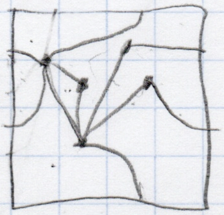
$V=5, E=9, F=4$

$V-E+F=0$



$V=4, E=6, F=2$

$V-E+F=0$



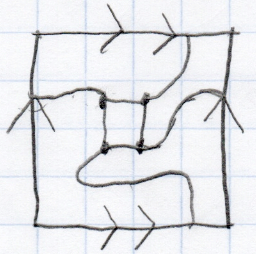
$V=5, E=8, F=3$

$V-E+F=0$

The Euler characteristic appears to be 0.

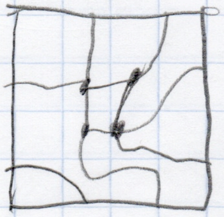
16.6: What we need to check:

- Adding an edge between two vertices
- Adding a separate edge and "leaf" vertice
- Removing either of these kinds of edges
- That each of these work using the torus (i.e. no planar connections added).



Original: $V=4$, $E=6$, $F=2$

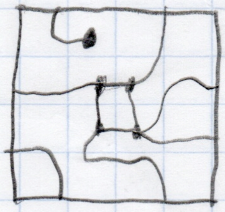
$V-E+F=0$



Vertice -> Vertice Edge:

$V=4$, $E=7$, $F=3$

$V-E+F=0$



Vertice -> Leaf Edge:

$V=5$, $E=7$, $F=2$

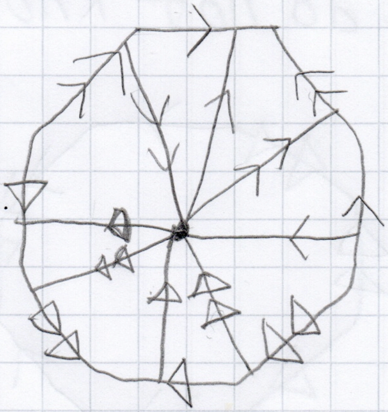
$V-E+F=0$

Removal simply goes back to the original. (can't remove from original, would create a "bad graph")

16.7: In a k_8, $V=8$ and $E=28$. $V-E+F=0$, so F must be 20. Because each face must have at least 3 edges, there should be at least $\frac{3F}{2}=30$ total edges. However, 28 is not at least 30. Contradiction.

*These represent the direction of all 8 sides.

16.8: The most minimal "good graph":

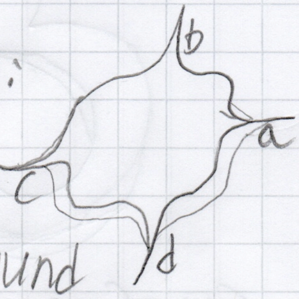


1 vertice, 4 edges, and (suprisingly) 1 face. V-E+F =-2. Any additional leaves or connected edges will leave V-E+F the same, as the torus.

So the Euler characteristic is -2.

16.9: Here are some possibilities:

- The two-holed torus (from 16.2; made with an $aba^{-1}b^{-1}cdc^{-1}d^{-1}$* pattern)
- A 4d pinched off-sphere made with an $aa^{-1}bb^{-1}cc^{-1}dd^{-1}$ pattern. Image of shape in 3D before the final fold step:

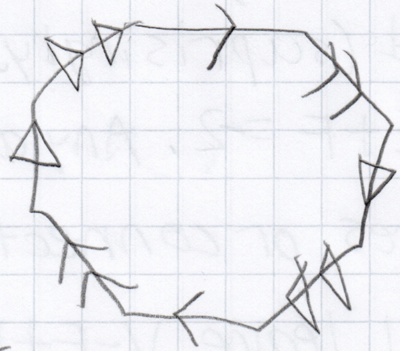


(Each pinched segment has to fold back around the sphere onto itself to finish.)

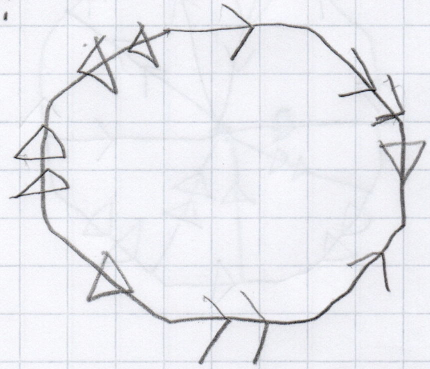
- A torus attatched to a klein bottle made with an $aba^{-1}b^{-1}cdCd$ pattern.
- A double-klein bottle: abab cdCd. (cont.)

*Also the number of topological holes.

16.9: Some configurations I do not know the final form of:



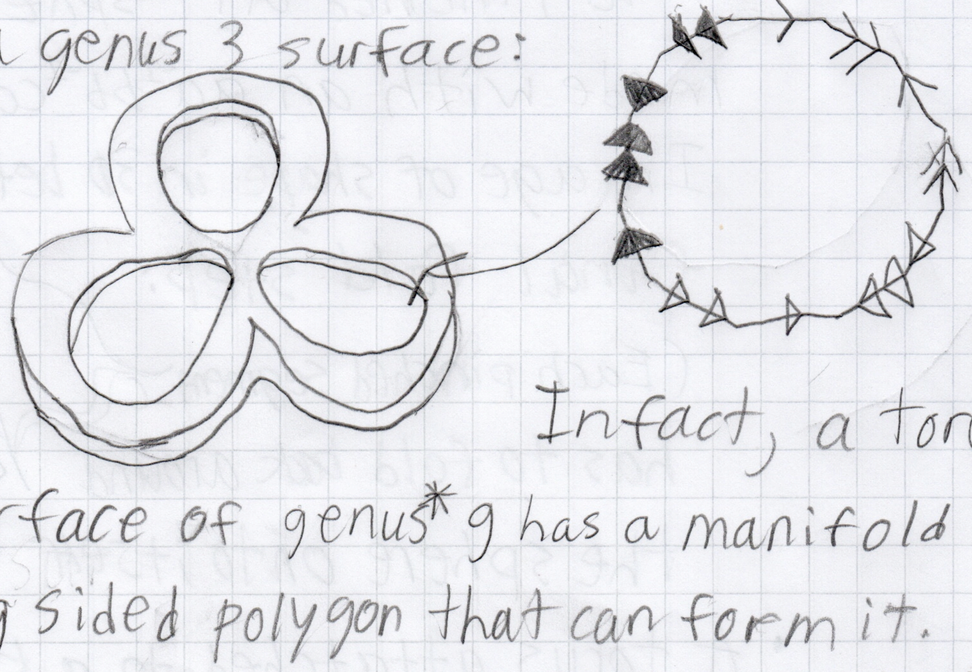
Four concurrent planes? Or another torus?



Two interlocking Klein bottles?

These were the notable ones I found. Any else?

16.10: Just as a rectangle and an octagon form a genus 1 and 2 surface, a dodecagon with $aba^{-1}b^{-1}cdc^{-1}d^{-1}efe^{-1}f^{-1}$ forms a genus 3 surface:



In fact, a toroidal surface of genus* g has a manifold of a $4g$ sided polygon that can form it. For $V-E+F$, $g=0 \rightarrow 2$, $g=1 \rightarrow 0$, $g=2 \rightarrow -2$. This forms a pattern of $V-E+F=2-2g$ as the Euler characteristic of genus g.