WORKSHEET 1

Fibonacci Numbers

The *Fibonacci numbers* are the sequence of numbers starting

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Each one is the sum of the previous two numbers. The sequence starts with the 0^{th} Fibonacci number, and we write F_n for the n^{th} Fibonacci number. Thus we have $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, and so on. The rule that each one is the sum of the previous two can be expressed by saying that

$$F_n = F_{n-1} + F_{n-2} \tag{1.1}$$

for all $n \geq 2$.

Problem 1.1. What is the first Fibonacci number that is greater than 1000?

Problem 1.2. If you know F_n and F_{n+1}, how do you use the rule (1.1) to compute F_{n-1}? Compute F_{-n} for $1 \leq n \leq 10$. What do you notice?

Problem 1.3. For which values of n is F_n even? Can you explain why?

Problem 1.4. For which values of n is F_n a multiple of 3? Can you explain why?

Problem 1.5. Show that for every positive integer m, there are infinitely many Fibonacci numbers that are divisible by m.

The *Lucas numbers* are defined similarly to the Fibonacci numbers, with each Lucas number being the sum of the previous two. However, the initial conditions are different: we have $L_0 = 2$ and $L_1 = 1$. The first few Lucas numbers, starting with L_0, are

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$$

Problem 1.6. Show that no Lucas number is divisible by 5. What makes the Lucas numbers behave differently from the Fibonacci numbers, as in problem 1.5?

PROBLEM 1.7. How many ways are there to tile a $1 \times n$ rectangle using 1×1 and 1×2 rectangles? All the possibilities for $n = 4$ are shown in Figure 1. (Start by making a table of the answers for small values of n until you recognize the pattern.)

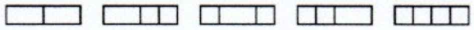


Figure 1. All ways of tiling a 1×4 rectangle with 1×1 and 1×2 rectangles.

PROBLEM 1.8. How many sequences of 0's and 1's are there of length n, such that there are no two consecutive 1's? When $n = 4$, these sequences are

$$0000, 0001, 0010, 0100, 0101, 1000, 1001, 1010.$$

PROBLEM 1.9. For $1 \leq n \leq 10$, compute $\frac{F_{n+1}}{F_n}$. What is going on with these ratios? Do they appear to be approaching a limit? Can you explain why? What about $\frac{L_{n+1}}{L_n}$, the ratio of consecutive Lucas numbers?

PROBLEM 1.10. Let $G_n = F_1 + F_2 + F_3 + \cdots + F_n$. Can you find a formula for G_n? Explain why this formula works. What about $M_n = L_1 + L_2 + L_3 + \cdots + L_n$?

PROBLEM 1.11. What if we add up only every other Fibonacci number? That is, let $H_n = F_1 + F_3 + F_5 + \cdots + F_{2n-1}$. What is H_n?

PROBLEM 1.12. What about $F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2$?

PROBLEM 1.13. What about $F_{n-1}F_{n+1} - F_n^2$?

PROBLEM 1.14. Explain what's going on in Figure 2: where did the missing square go?

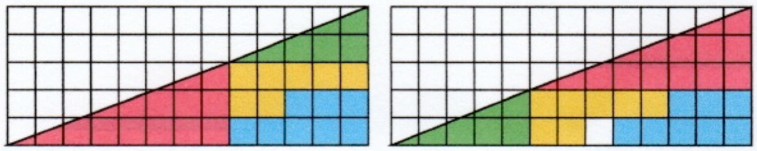


Figure 2. Where did the missing square go?

Euler Circle HW 1: Fibbonacci Numbers

Alexander F. Sept. 23-30, 2025

1.1: List out the Fibbonacci numbers, starting from 0:
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,
233, 377, 610, 987 (almost), 1597
is the first Fibbonacci number >1,000.

1.2: Because $F_n = F_{n-1} + F_{n-2}$, moving the F_{n-2} term gives $F_{n-1} = F_n - F_{n-2}$. This equation helps us find negative "Fibbonaccies":
Starting with $F_{n-1} = 0 = F_0$, $F_{-1} = F_{n-2} = -1$.
Then, listing them all down to F_{-10} gives...
... -1, 1, -2, 3, -5, 8, -13, 21, -34, 55.
They are the same as positive Fibbonacci numbers, but they switch signs (+/-).

1.3: Fibonacci numbers alternate between
one (odd F_{n-1}) + (odd F_{n-2}) = (even F_n)
and
two (odd $F_{n-1/n-2}$) + (even $F_{n-1/n-2}$) = (odd F_n)s.
Therefore, for every third n, these equations add up to an odd F_n. This means that n is a multiple of 3.

*As long as their factors do, they must.

1.4: This is similar to 1.3, but write a list:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,

F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10}, F_{11}, F_{12},

It is evident via this list that there is an infinite pattern of having a multiple of 3 every multiple of 4 of n (in F_n).

1.5: Let's assume there is an m where it is divisible by a finite number of Fibbonaci numbers and try to find it.

- $m=1$: This always works!
- $m=2$: This works, see 1.3.
- $m=3$: This also works, see 1.4.
- $m=4$: Every 6th Fibbonnaci number works.
- $m=5$: Every 5th Fibbonnaci number works.
- (Skip numbers that aren't prime from now on, they repeat*)
- $m=7$: Every 8th Fibbonacci number works.
- $m=11$: Every 10th Fibbonacci number works.
- $m=13$: Every 7th ...
- $m=17$: Every 9th...

It seems like <u>forever</u>... Because it is! Using the same idea as 1.6 and 1.4, each m gives a pattern of remainders when the whole Fibbonaci sequence is divided by it. These patterns ALWAYS loop back to 0, giving ∞.

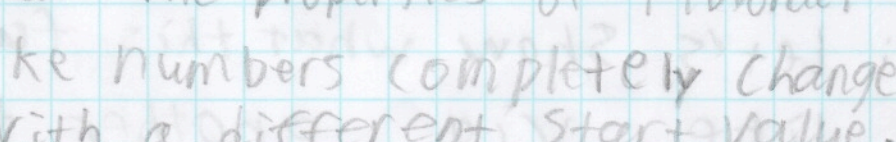
1.6: To see why no Lucas number is divisible by 5, let's list a few:

L. Number	2	1	3	4	7	11	18	29	47	76	123	199
Remainder when ÷5	2	1	3	4	2	1	3	4	2	1	3	4
L_n Notation	L_0	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}

The remainders when ÷5 are never =0... And because each number is the sum of the previous two, it never will be. This shows that the properties of Fibbonaci-like numbers completely change with a different start value, as in not having an infinite multiple of 5 (or any 10, 15, 20...).

1.7: Again, find the pattern and make a table:

n	1	2	3	4	5	6	7
ways	1	2	3	5	8	13	21



The number of ways is always F_{n+1}. Interesting...

1.8: Same approach as 1.7:

n	1	2	3	4	5	6
ways	2	3	5	8	13	21
	0 1	00 01 10	000 100 001 010 101	0000 0001 0010 0100 1000 1001 1010 0101	...	...

Here, the number of ways is always F_{n+2}! Suprising how Fibbonacci numbers sneak into problems like these...

1.9: Let's show what this function looks like via another table:

n	1	2	3	4	5	6	7	8	9	10
$\frac{F_{n+1}}{F_n}$	2	2	1.5	$1.\overline{333}$...	1.2	1.125	1.076 92307 7...	1.0476 19048 ...	1.0294 11765 ...	1.018...

Starting from $\frac{1+1}{1} = 2$, the ratio slowly approaches 1, but never gets to 1.

Now for the Lucas numbers:

n	1	2	3	4	5	6	7	8	9	10
$\frac{L_{n+1}}{L_n}$	2	$1.\overline{333}$...	1.25	1.1428 57143 ...	1.09...	1.055...	1.071 44827 59...	1.021 276596 ...	1.013 9583 05...	1.008130 081

Again, they start from 2, and again, they approach 1 but never reach it - just at a faster rate.

*I have said this too many times.

1.10: Again, patterns reveal clues*:

n	1	2	3	4	5	6	7	8	9	10
G_n	1	2	4	7	12	20	33	54	88	143

All outputs are $F_{\text{something}}-1$, and all outputs are two fibbonacci numbers ahead of F_n, minus one. This gives $G_n = F_{n+2}-1$.

But wait! what about Lucas numbers?

n	1	2	3	4	5	6	7	8	9	10
M_n	1	4	8	15	26	44	73	120	196	319

Here, all outputs are $L_{\text{something}}-3$, and all outputs are two Lucas numbers ahead of L_n, minus three. This gives $M_n = L_{n+2}-3$.

1.11:

n	1	2	3	4	5	6	7	8	9	10
H_n	1	3	8	22	55	144	377	987	2584	6765
Relation to F_n	F_n	F_{n+2}	F_{n+3}	F_{n+4}	F_{n+5}	F_{n+6}	F_{n+7}	F_{n+8}	F_{n+9}	F_{n+10}

There is a complicated way to use the third row, but notice that every H_n is an even Fibbonacci number, more specifically F_{2n}? Therefore, $H_n = F_{2n}$.

1.12:

n	1	2	3	4	5	6	7	8	9	10
$F_1^2+F_2^2+\dots F_n^2$	1	2	6	15	40	104	273	714	1870	4895

Because we are sauaring here, it's reasonable to look for products. And, lo and behold:

$F_1 \cdot F_2 = 1 \cdot 1 = 1 = F_1^2$.

$F_2 \cdot F_3 = 1 \cdot 2 = 2 = F_1^2 + F_2^2$.

$F_3 \cdot F_4 = 2 \cdot 3 = 6 = F_1^2 + F_2^2 + F_3^2$.

This pattern continues forever (or at least to 10.) Whenever two consecutive Fibonacci numbers are multiplied, the result is the square of the first with all of the proceding summed, like $F_n \cdot F_{n+1} = F_n^2 + F_{n-1}^2 + \dots$

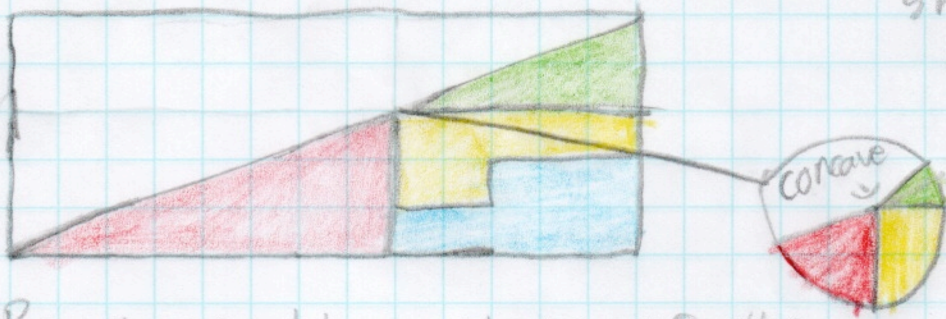
1.13: Final problem with a table:

n	1	2	3	4	5	6	7	8	9	10
$F_{n-1} \cdot F_{n+1} - F_n^2$	-1	1	-1	1	-1	1	-1	1	-1	1

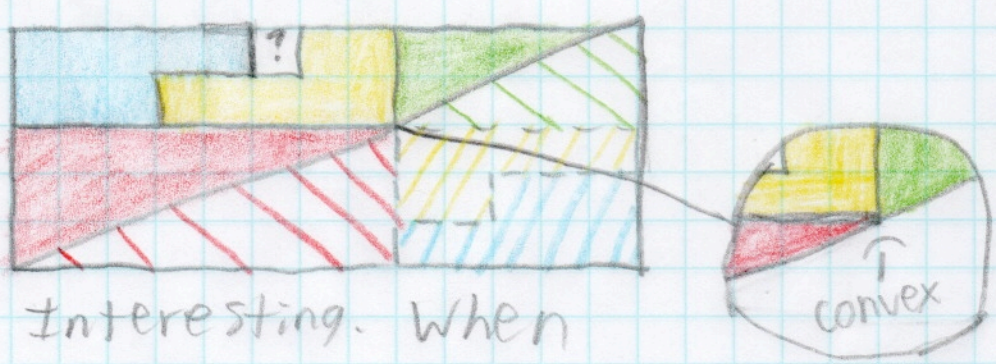
Interesting... Whenever n is even, the result is 1, and whenever odd the result is -1. This can be simply expressed as $(-1)^n$ because even powers give positive 1, while odd powers give negative 1.

*colored "blocks," 2 triangles + 2 rectangular-shaped hexagons

1.14:



Because the slope of the green triangle $\neq$ the slope of the red triangle, ($\frac{2}{5} \neq \frac{3}{8}$), the colored area isn't $\frac{1}{2}$. If we attempt to move every piece* to fill up a bit <u>more</u> than $\frac{1}{2}$ of the area, we get:



Interesting. When these shapes attempt to form more than $\frac{1}{2}$ the rectangle's area, the missing square shows the slight difference in slope between the red and green triangles to the naked eye.