WORKSHEET 9

Counting Using Binomial Coefficients

We met the binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ in the previous worksheet, thanks to their appearance in Pascal's triangle. But they also show up in many other places.

PROBLEM 9.1. How many ways are there to select a k-member committee from among n people?

PROBLEM 9.2. Let m and n be positive integers. How many paths are there from $(0,0)$ to (m,n) taking steps that go either one unit to the right or one unit up?

PROBLEM 9.3. How many ways are there to choose a committee (of any size) from among n people, then choosing one member of the committee to be the president? Can you solve this problem in two different ways and obtain an interesting identity involving binomial coefficients?

We now turn to a very important technique involving binomial coefficients, used to solve problems that initially do not appear to be well-suited to binomial coefficients. This method, called the *stars and bars technique*, involves setting up several objects, which are initially vertical bars, and then converting some of the bars into stars. Another variation involves setting up a bunch of stars, then placing bars in some of the gaps between consecutive stars.

PROBLEM 9.4. Suppose we start with n bars in a row. How many ways are there to convert k of the bars into stars?

PROBLEM 9.5. Suppose we start with n stars in a row. How many ways are there to place k bars in some of the gaps between adjacent stars? Each gap is only allowed to contain at most one bar.

PROBLEM 9.6. How many ways are there to write a positive integer n as a sum of k positive integers, where order matters? For instance, the representations $1+1+3$ and $1+3+1$ for 5 are considered to be different representations.

PROBLEM 9.7. A bagel shop sells 8 different types of bagels. How many ways are there to select a box of a baker's dozen (i.e. 13) bagels?

Two boxes are considered the same if they contain the same number of each type of bagel.

PROBLEM 9.8. How many ways are there to write a nonnegative integer n as a sum of k nonnegative integers, where order matters?

PROBLEM 9.9. How many n-digit numbers are there whose digits are strictly increasing from left to right?

PROBLEM 9.10. How many n-digit numbers are there whose digits are nondecreasing from left to right? (In other words, this is the same as the previous problem, but now consecutive digits can be equal.)

PROBLEM 9.11. Suppose you have m cats and n dogs. How many ways are there to select a total of r animals? Can you solve this problem in two ways and obtain an interesting identity involving binomial coefficients?

PROBLEM 9.12. Can you express the sum

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2$$

in closed form, i.e. without using a sum over a variable number of terms?

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9.1: There are $\binom{n}{k}$ ways. In Pascal's Δ we remember that every term can be expressed as $\binom{n}{k}$ and as the number of ways to get to itself from the apex - in other words, the number of ways to choose the term's index out of the row's index, expressable as $\binom{n}{k}$.

9.2: This, of course, is simply 8.9 but you tilt your head (look at my 8.9 diagram) and see the Pascal's Triangle inside. Of course, every term can be $\binom{n}{k}$, so (m, n) is $\binom{m}{n}$ paths.

9.3: Assuming the committee is size k:

Way 1: Choose the committee then the president.

$\binom{n}{k}$, then $\binom{k}{1}$. $\frac{n!}{(n-k)!\cancel{(k)}!} \cdot \frac{\cancel{k!}}{(k-1)!} = \frac{n!}{(n-k)!(k-1)!}$

Multiply both sides by k: $\frac{k \cdot n!}{(n-k)!k!} = \boxed{k \cdot \binom{n}{k}}$

Way 2: Choose the president then the committee.

$\binom{n}{1}$, then $\binom{n-1}{k-1}$. $\frac{n!}{\cancel{(n-1)!}} \cdot \frac{\cancel{(n-1)!}}{(n-k)!(k-1)!} = \frac{n!}{(n-k)!(k-1)!}$

Same, $\cdot \frac{k}{k} = \frac{k \cdot n!}{(n-k)!k!} = \boxed{k \cdot \binom{n}{k}}$

9.4: This is essentially the committee problem, so the number of ways is $\binom{n}{k}$.

9.5: In this problem, there are $n-1$ gaps for n stars, and therefore $n-1$ places to put a total of k bars. So this is $\binom{n-1}{k}$.

9.6: This is also $\binom{n-1}{k-1}$ ways! For example:

$5 = 1+1+3 = 1+3+1 = 3+1+1 = 2+2+1 = 2+1+2 = 1+2+2$.

6 ways to express 5 as 3 positive integers - but six is also $\binom{4}{2} = \binom{5-1}{3-1}$.

Not sure on the proof, though.

9.7: Obviously, $\binom{8}{13}$ isn't possible on its own. But "stars and bars" help. With 13 bagels and 7 bars to split them into 8 categories, counting is simpler. There is a total of 26 slots for each bar (some bars can be outside the stars), so $\binom{26}{7}$ is a whopping 657,800 possible boxes of 13 bagels (calculator).

9.8: Testing $n=5$ and $k=3$ gives $\binom{n+2}{k}$.

Not sure how to prove again, though.

*#=short for "number"

9.9: The answer turns out to be in the 10th row of Pascal's Triangle:

1 9 36 84 126 126 84 36 9 1

There is 1 way to arrange a 0-digit #*, 9 such 1-digit #s, 36 possible for 2 digits, 84 for 3-digit #s, 126 for 4 and 5-digit #s, 84 6-digit #s, 36 7-digit #s, 9 8-digit #s, and only 1 9-digit # (123456789). No 10-digit number cannot repeat a digit, so 123456789 is the biggest case. The 10th row of Pascal's Δ has index 9, and here there are 9 available digits (0 is impossible), so for each $\binom{9}{n}$ in Pascal's Δ results the number of possible, well, numbers.

9.10: It turns out "stars and bars" helps:

$\bullet\ \bullet\ \bullet \ldots$ and $|\ |\ |\ |\ |\ |\ |\ |$.

8 bars divides n stars into 9 "digit groups", which are arranged into one unique number. n+8 gaps gives $\binom{n+8}{8}$ ways.

9.11: Method 1: Use "Stars and Bars."
Say there are r stars, and
1 bar dividing r into cat-stars and
dog-stars. That leaves n+1 possible
positions to place the bar, instantly
dividing r into chosen cats and chosen
dogs. This already satisfies the restriction,
So there are <u>r+1</u> ways to choose r animals.
Method 2: Go through each possibility.
Say that x is the number of chosen cats,
and y likewise for dogs. So $x+y=r$.
Now write x and y in terms of r:

$x=0$, $y=r$. $0+r=r$.

$x=1$, $y=r-1$. $1+r-1=r$.

$x=2$, $y=r-2$. $2+r-2=r$.

. . . .

$x=r$, $y=0$. $r+0=r$.

The total number of ways to write
these cases is the number of terms in the set
[0,1,2,...r], which is also <u>r+1</u>.

9.12: This equation is also just every term of row n in Pascal's Δ, squared then summed. It turns out that this sum always equals the middle term of the 2nth row:

$1^2+2^2+1^2=6$ 2nd row.

4th row: 1 4 6 4 1.

$1^2+3^2+3^2+1^2=20$ 3rd row.

6th row: 1 6 15 20 15 6 1

In other words, this can be expressed simply as $\binom{2n}{2n/2}$, which is just $\binom{2n}{n}$.

NOTE: I'm not sure on the proof of the generalization of this pattern. Any ideas?