WORKSHEET 18

Vieta's Formulae

There are two basic ways we can represent a polynomial: either in terms of its coefficients, or in terms of its roots. That is, if we have a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, with complex roots $r_1, r_2, \ldots, r_n$, then we can factor it as $f(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n)$. Note that f might have fewer than n complex roots, because some of them could be repeated, such as in the case of $f(x) = x^2 - 2x + 1 = (x-1)^2$. We need to count each root with the correct multiplicity.

Since we have two representations for the same polynomial, there ought to be some relations between the two of them.

PROBLEM 18.1. Suppose $f(x) = x^2 + ax + b = (x - r_1)(x - r_2)$. Express a and b in terms of r_1 and r_2.

PROBLEM 18.2. Suppose $f(x) = x^3 + ax^2 + bx + c = (x - r_1)(x - r_2)(x - r_3)$. Express a, b, and c in terms of r_1, r_2, and r_3.

PROBLEM 18.3. More generally, suppose $f(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = (x - r_1)(x - r_2)\cdots(x - r_n)$. Express the a_i's in terms of the r_j's.

PROBLEM 18.4. Finally, do the general case, where $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = a_n(x - r_1)(x - r_2)\cdots(x - r_n)$. The formulae you should have ended up with are called *Vieta's formulae.*

As you can see, the case where the leading coefficient (a_n) is equal to 1 can be used to do the general case and leads to slightly cleaner formulae, so from now on we'll just assume that $a_n = 1$ unless otherwise noted. A polynomial with leading coefficient equal to 1 is said to be a *monic polynomial.*

PROBLEM 18.5. Let $f(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ have roots $r_1, r_2, \ldots, r_n$. Express $r_1^2 + r_2^2 + \cdots + r_n^2$ in terms of the a_i's.

PROBLEM 18.6. With notation as in the previous problem, express $\frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_n}$ in terms of the a_i's.

PROBLEM 18.7. Let $f(x) = x^3 + ax^2 + bx + c$ have roots α, β, γ. Find a polynomial whose roots are $\alpha + 2, \beta + 2, \gamma + 2$, in terms of a, b, c.

PROBLEM 18.8. With notation as in the previous problem, find a polynomial whose roots are $\alpha^2, \beta^2, \gamma^2$.

PROBLEM 18.9. With notation as in the last two problems, find a polynomial whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$. (Assume that none of α, β, γ are 0.

PROBLEM 18.10. With notation as in the last three problems, find a polynomial whose roots are $\alpha\beta, \beta\gamma, \alpha\gamma$.

PROBLEM 18.11. Solve the system of equations

$$\begin{aligned} \alpha + \beta + \gamma &= 17, \\ \alpha\beta + \beta\gamma + \alpha\gamma &= 92, \\ \alpha\beta\gamma &= 160. \end{aligned}$$

‡Factored amongst all x coefficients multiplied!

Vieta's Formulae

*or r_1+r_2, or nr_1+mr_2

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18.1: $a=-r_1-r_2$*, $b=r_1\cdot r_2$. This is what the coefficients form after a standard quadratic expansion.

18.2: $a=-r_1-r_2-r_3$, $b=r_1r_2+r_2r_3+r_3r_1$, $c=-r_1r_2r_3$. Again, standard cubic expansion pairs roots into these coefficients.

18.3: The pattern of coefficients continues:

$a_{n-1}=-r_1-r_2-r_3\ldots-r_n$,

$a_{n-2}=r_1r_2+r_1r_3\ldots+r_2r_3+r_2r_4\ldots+r_{n-1}r_n$,

$a_{n-3}=-r_1r_2r_3-r_1r_3r_4\ldots-r_2r_3r_4\ldots-r_{n-2}r_{n-1}r_n$,

$\ldots a_0=\pm r_1r_2r_3r_4r_5r_6r_7r_8\ldots r_{n-1}r_n$.

18.4: Instead of writing a new formula, divide both sides by a_n:

$a_nx^n+a_{n-1}x^{n-1}\ldots a_1x+a_0=a_n^{‡}(x-r_1)(x-r_2)\ldots(x-r_n)$,

becomes $x^n+\frac{a_{n-1}x^{n-1}}{a_n}\ldots+\frac{a_0}{a_n}=(x-r_1)(x-r_2)\ldots(x-r_n)$.

This has the starting x^n and the starting roots, so the Vietas can be derived with a small change:

$\frac{a_{n-1}}{a_n}=-r_1-r_2-r_3\ldots-r_n$, $\frac{a_{n-2}}{a_n}=r_1r_2+r_1r_3\ldots+r_2r_3\ldots+r_{n-1}r_n$. $\frac{a_{n-3}}{a_n}=-r_1r_2r_3\ldots-r_{n-2}r_{n-1}r_n\ldots\frac{a_0}{a_n}=\pm r_1r_2r_3\ldots r_n$.

Same formulae, but all over a_n to account for generality.

*The negative comes from an odd degree of multiplied roots making up the coefficient a_1 itself. ‡I got tired of Greek letters, so now a', b', c'! prime

18.5: Note that $(r_1+r_2\dots+r_n)^2=(r_1^2+r_2^2\dots+r_n^2)+\dots$

$\dots 2(r_1r_2+r_1r_3\dots+r_2r_3\dots+r_{n-1}r_n)$, because of how multinomials distribute into a combination of squares and products of 2 different roots.

This equation can be rewritten to:

$r_1^2+r_2^2\dots+r_n^2=(\text{sum of roots})^2+2(r_1r_2+r_1r_3\dots+r_{n-1}r_n)$.

From 18.4 the first term right side is $\left(\frac{a_{n-1}}{a_n}\right)^2$ and the second term right side is $2\left(\frac{a_{n-2}}{a_n}\right)$.

So, $r_1^2+r_2^2,\dots+r_n^2=\left(\frac{a_{n-1}}{a_n}\right)^2+\left(\frac{a_{n-2}}{a_n}\right)\cdot 2=\boxed{\frac{a_{n-1}^2-2\cdot a_n\cdot a_{n-2}}{a_n^2}}$.

18.6: $\frac{1}{r_1}+\frac{1}{r_2}+\dots+\frac{1}{r_n}$ as one fraction has a denominator $r_1\cdot r_2\cdot\dots\cdot r_n$ and a numerator $r_1r_2r_3\dots r_{n-1}+r_1r_2\dots r_{n-2}r_n\dots+r_2r_3\dots r_n$.

These are respectively represented by the Vieta coefficients $\frac{a_0}{a_n}$ and $-\frac{a_1}{a_n}$*.

So, the final sum is $\frac{\frac{a_0}{a_n}}{-\frac{a_1}{a_n}}=\frac{a_0}{-a_1}=\boxed{-\frac{a_0}{a_1}}$.

18.7: Write out new a,b,c in terms of roots:

$a=\alpha+\beta+\gamma+6$, $b\overset{‡}{=}(a'+2)(b'+2)+(b'+2)(c'+2)+(c'+2)+(a'+2)=\dots$

$\dots=a'b'+2a'+2b'+4+b'c'+2b'+2c'+4+a'c'+2a'+2c'+4=a'b'+b'c'+a'c'+4a'+4b'+4c'+12$, $c=(a'+2)(b'+2)(c'+2)=a'b'c'+2a'b'+2a'c'+2b'c'\dots$

$\dots+4b'+4a'+4c'+8$. Simplifying these gives... (cont.

*Interesting, a reciprocal of a polynomial over c...

18.7 (cont.)... new $a = a+6$, new $b = b+4a+12$,

new $c = c+2b+4a+8$. Substitute in polynomial:

$$\boxed{\begin{aligned}&x^3+(a+6)x^2+(b+4a+12)x+c+2b+4a+8.\\&=x^3+ax^2+6x^2+bx+4ax+12x+c+2b+4a+8.\end{aligned}}$$

(Whichever answer you prefer.)

18.8: Again, same thing: new $a = a'^2+b'^2+c'^2$, new $b = a'^2b'^2+b'^2c'^2+c'^2a'^2$, new $c = a'^2b'^2c'^2$. Rewrite:

new $a = (a'+b'+c')^2-2ab-2ac-2bc = a^2-2b$. new $b = (a'b'+b'c'+a'c')^2-2a'b'^2c'-2a'b'c'^2-2a'^2b'c' = (a'b'+b'c'+a'c')^2-2a'b'c'(a'+b'+c') = b^2-2ca$. new $c = (a'b'c')a'b'c'$...

...$= c^2$! Substitute: $\boxed{x^3+(a^2-2b)x^2+(b^2-2ca)x+c^2.}$

$\boxed{=x^3+a^2x^2-2bx^2+b^2x-2cax+c^2.}$ Again, both ways work.

18.9: Here we go! new $a = \frac{1}{a'}+\frac{1}{b'}+\frac{1}{c'} = \frac{b'c'+a'c'+a'b'}{a'b'c'}$. new $b = \frac{1}{a'}\cdot\frac{1}{b'}+\frac{1}{b'}\cdot\frac{1}{c'}+\frac{1}{c'}\cdot\frac{1}{a'} = \frac{1}{a'b'}+\frac{1}{b'c'}+\frac{1}{c'a'} = \frac{c'+a'+b'}{a'b'c'}$.

new $c = \frac{1}{a'}\cdot\frac{1}{b'}\cdot\frac{1}{c'} = \frac{1}{a'b'c'}$. Rewrite each to coefficients:

new $a = \frac{b}{c}$, new $b = \frac{a}{c}$, new $c = \frac{1}{c}$. Substitute:

$$\boxed{x^3+\frac{bx^2}{c}+\frac{ax}{c}+\frac{1}{c} = \frac{cx^3+bx^2+ax+1}{c}^{*}.}$$

18.10: Once more! new $a = a'b'+b'c'+c'a'$, new $b = a'b'^2c'+a'b'c'^2+a'^2+b'+c' = a'b'c'(a'+b'+c')$. new $c = a'^2b'^2c'^2$. Rewrite:

new $a = b$, new $b = ac$, new $c = (a'b'c')a'b'c' = c^2$!

Substitute: $\boxed{x^3+bx^2+acx+c^2.}$ The nicest yet!

18.11: Instead of painfully solving it, look at each equation as a Vieta cubic coefficient: where $a=17$, $b=92$, and $c=160$. Solving the resulting cubic will yield the 3 roots that in turn solve the system. So:

$x^3+17x^2+92x+160=0$. -8 is a factor that substitutes to 0, so $(x+8)$ is one Part of the trinomial. Dividing $x^3+17x^2+92x+160$ by $(x+8)$ synthetically gives the quadratic $x^2+9x+20$. Using the Quad. Formula,

$$x=\frac{-9\pm\sqrt{81-80}}{2}=\frac{-9\pm1}{2}=-4,-5.$$

So factoring of the cubic is $(x+8)(x+4)(x+5)$, and the roots and solutions to the system are -4, -5, and -8.

(Note: The actual solutions are positive 4, 5, and 8, but the initial quadratic created happened to yield negative roots.)